

Analytic knots, satellites and the 4-ball genus

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Abstract

Call a smooth knot (or smooth link) in the unit sphere in \mathbb{C}^2 analytic (respectively, smoothly analytic) if it bounds a complex curve (respectively, a smooth complex curve) in the complex ball. Let K be a smoothly analytic knot. For a small tubular neighbourhood of K we give a sharp lower bound for the 4-ball genus of analytic links L contained in it.

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1 Introduction

Let K be a smooth knot in the 3-sphere S^3 . A smooth knot or smooth link L contained in a tubular neighbourhood $N(K) \subset S^3$ of K is a satellite of K if it is not isotopic to K in $N(K)$ and (any connected component of L) is not contained in a 3-ball inside $N(K)$. Satellites of non-trivial knots have been considered since long. They play a role in the problem which knot complements admit hyperbolic structure and received recent interest from the point of view of invariants of knots and links. In the following we always consider knots and links to be smooth and oriented. Speaking about tubular neighbourhoods we will always consider them smoothly bounded. Let K be a non-trivial knot and L a link contained in a smoothly bounded tubular neighbourhood $N(K)$ of K . Define an entire number n in the following way. Consider a projection $\text{pr} : N(K) \rightarrow K$. The image $\text{pr}(L) \subset K$ is homologous to $n \cdot [K]$ (in $H_1(K)$) for an entire number n . Write $w_{N(K)}(L) = n$ and call n the winding number of L in $N(K)$. (Having in mind a specified tubular neighbourhood we will also speak about the winding number of L around K .)

The pattern \mathcal{L} of L gives more precise information on the satellite L . It is defined as follows. Denote by U a standard realization of the unknot. For instance, identify S^3 with the unit sphere $\partial\mathbb{B}^2$ in \mathbb{C}^2 . Let $H = \{z_2 = 0\}$ be the first coordinate line in \mathbb{C}^2 and let U be the unknot $U = \partial\mathbb{B}^2 \cap H$ oriented as boundary of a complex disc in \mathbb{B}^2 . Consider a tubular neighbourhood $N(U) \subset S^3$ of U . Trivialise both, $N(K)$ and $N(U)$, by Seifert framing, i.e. by a transversal vector field on the knot which points in the direction of a smooth oriented surface which is contained in the sphere and bounded by the knot. Such a surface is called a Seifert surface. Note that the trivialization does not depend on the choice of the Seifert surface. Consider a diffeomorphism $\varphi_K : N(K) \rightarrow N(U)$ which maps Seifert framing to Seifert framing. The pattern \mathcal{L} of the satellite L is the isotopy class of $\varphi_K(L)$ in $N(U)$.

A classical paper of Schubert [18] relates the genus of a knot to the genus of its satellites. The (smooth) genus $g(K)$ of a knot (or link) K is the minimal genus among smooth oriented surfaces in S^3 bounded by K . (If L is a link and the surface is not connected we mean the sum of the genera of the connected components.) The genus depends only on the isotopy class of the knot or link. Define the genus of an isotopy class of links as the genus of its representatives. Schubert's theorem is the following.

For a satellite knot L in a tubular neighbourhood $N(K)$ of a knot $K \subset S^3$ with $n = w_{N(K)}(L)$ the following inequality for the genera holds

$$g(L) \geq |n| g(K). \quad (1)$$

Moreover,

$$g(L) \geq |n| g(K) + g(\mathcal{L}). \quad (2)$$

Identify again S^3 with the unit sphere $\partial\mathbb{B}^2$ in \mathbb{C}^2 . We consider links (or knots) which are obtained as the transverse intersection of $\partial\mathbb{B}^2$ with a relatively closed complex curve \tilde{X} in a neighbourhood of the closed unit ball \mathbb{B}^2 . Following Rudolph [16] we call such links analytic, and smoothly analytic if the complex curve $\tilde{X} \cap \mathbb{B}^2$ bounded by the link is smooth (i.e., non-singular). We always consider an analytic link in the sphere $\partial\mathbb{B}^2$ oriented as boundary of a complex curve in the unit ball.

Since $H^2(\mathbb{B}^2, \mathbb{Z}) = 0$ the complex curve is the zero locus $\{z \in \mathbb{B}^2 : f(z) = 0\}$ of an analytic function in a neighbourhood of the closed ball. The function f can be uniformly approximated on \mathbb{B}^2 by a polynomial, which gives an isotopic link in $\partial\mathbb{B}^2$ that bounds a piece of an algebraic hypersurface. If the curve $\{z \in \mathbb{B}^2 : f(z) = 0\}$ is singular its genus is defined to be the genus of its smooth perturbation $\{z \in \mathbb{B}^2 : f(z) = \varepsilon\}$ for generic small enough numbers ε .

We are interested in the (smooth) 4-ball genus $g_4(L)$ of a knot (or link) L , called also slice genus. This is the minimal genus among smooth oriented surfaces embedded into \mathbb{B}^2 and bounded by L . Always $g_4(L) \leq g(L)$ but $g_4(L)$ may be strictly smaller than $g(L)$. The 4-ball genus gives a lower bound for the unknotting number of a knot, the smallest number of crossing changes needed to unknot the knot. The class of analytic knots is interesting from the point of view of knot invariants: for them half the Rasmussen invariant and also the τ -invariant are equal to the 4-ball genus of the knot ([15], [4], [19]).

By a consequence of a deep theorem of Kronheimer and Mrowka (Corollary 1.3 of [11], the local Thom Conjecture) the 4-ball genus of an analytic knot is realized on the complex curve bounded by it. The proof of Kronheimer and Mrowka also shows that for a link which bounds a connected complex curve in \mathbb{B}^2 its 4-ball genus is realized by the genus of this curve.

The following theorem holds.

Theorem 1. *Let K be a smoothly analytic knot in $\partial\mathbb{B}^2$. There exists a tubular neighbourhood $N(K) \subset \partial\mathbb{B}^2$ of K such that for any analytic link $L \subset N(K)$ the number $n = w_{N(K)}(L)$ is non-negative and the following statements hold.*

1. *If L is itself a knot then*

$$g_4(L) \geq ng_4(K) - \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (3)$$

($\lfloor x \rfloor$ denotes the largest integer not exceeding the real number x).

2. *Let L be a link which bounds a connected complex curve Y . If n is positive then the following lower bound for the 4-ball genus holds*

$$g_4(L) \geq ng_4(K) - (n-1). \quad (4)$$

For $n = 1$ the statements are true also if K bounds a singular curve.

The estimates are sharp in the following sense.

3. For each smoothly analytic knot K with $g_4(K) \geq 1$, each natural number $n \geq 1$ and any tubular neighbourhood $N(K) \subset \partial \mathbb{B}^2$ of K there exists a link $L \subset N(K)$ with $w_{N(K)}(L) = n$ which bounds a connected complex curve such that equality in (4) is attained.
4. Further, for each smoothly analytic knot K with $g_4(K) \geq 1$ and each natural number $n \geq 1$ there is a smoothly analytic knot K_1 which is smoothly isotopic to K and has the following property. For any tubular neighbourhood $N(K_1) \subset \partial \mathbb{B}^2$ of K_1 there exists an analytic knot $L \subset N(K_1)$ with winding number $w_{N(K)}(L) = n$ such that equality in (3) is attained for the knot K_1 and the knot L . In general, the original knot K does not have this property.

The condition of analyticity of K and L cannot be removed. Indeed, for any knot K (in particular, for an analytic knot K) and any tubular neighbourhood $N(K)$ the connected sum L with its mirror can be realized as a satellite of K with $n = w_{N(K)}(L) = 1$ and $g_4(L) = 0$.

We do not know how big the tubular neighbourhood $N(K)$ in the theorem can be chosen, in particular, we do not know whether for $n = 1$ the statement is true for analytic links in *any* tubular neighbourhood of an analytic knot. For $n > 1$ we do not know sharp estimates of the 4-ball genus of satellites of knots which bound singular complex curves.

Some satellites are especially simple and useful. They are defined in terms of closed braids. Recall the following definitions. Let $C_n(\mathbb{C}) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_i \neq z_j \text{ for } i \neq j\}$ be the configuration space of n particles which move in the plane without collision. The symmetrized configuration space $C_n(\mathbb{C})/\mathcal{S}_n$ is the quotient of $C_n(\mathbb{C})$ by the action of the symmetric group \mathcal{S}_n . Each point in $C_n(\mathbb{C})/\mathcal{S}_n$ can be considered as unordered tuple of n points and can be identified with the monic polynomial whose collection of zeros equals this unordered tuple. The space of monic polynomials of degree n without multiple zeros is denoted by \mathfrak{P}_n , the space of all monic polynomials of degree n is denoted by $\overline{\mathfrak{P}}_n$. The set of coefficients of polynomials in \mathfrak{P}_n is equal to $\mathbb{C}^n \setminus \{D_n = 0\}$, where D_n is the discriminant, i.e. D_n is a polynomial on \mathbb{C}^n which vanishes exactly if the monic polynomial with these coefficients has multiple zeros.

Recall that a geometric braid with base point $E_n \in \mathfrak{P}_n$ can be considered as a continuous map of the interval $[0, 1]$ into \mathfrak{P}_n with initial and terminating point equal to E_n . A braid with base point E_n is an isotopy class of geometric braids with this base point. Such braids form a group which is isomorphic to a group \mathcal{B}_n with $n - 1$ generators, denoted by $\sigma_1, \dots, \sigma_{n-1}$, and finitely many relations. The group is called Artin's braid group.

A braid is quasi-positive if it is the product of conjugates of the standard generators σ_i of \mathcal{B}_n (conjugates of inverses of these generators are not allowed as factors).

We call an oriented closed curve \tilde{L} in $S^1 \times D^2$ a closed geometric braid if the projection to S^1 is orientation preserving on \tilde{L} . The circle S^1 is assumed to be oriented, D^2 is a disc of real dimension 2. The number of preimages of

a point under the projection is called the number of strands. A closed braid is a free isotopy class of closed geometric braids. (Free isotopy means isotopy without fixing a base point.) It is well-known that free isotopy classes of closed geometric braids on n strands (for short, closed n -braids) are in one to one correspondence to conjugacy classes in Artin's braid group \mathcal{B}_n of braids on n strands (see e.g. [2]).

The notion of closed geometric braids is sometimes used in a more special situation, namely, by a closed geometric braid one means an oriented closed curve \tilde{L} in $\partial\mathbb{B}^2 \setminus \{z_1 = 0\}$ for which $d \arg z_1|_{\tilde{L}} > 0$.

Note that alternatively a geometric braid with base point E_n can be considered as a collection of n disjoint arcs in $[0, 1] \times D^2$ which join the collection $\{1\} \times E_n$ in the top $\{1\} \times D^2$ with the "identical" collection $\{0\} \times E_n$ in the bottom $\{0\} \times D^2$ and is such that for each arc the canonical projection to $[0, 1]$ is a homeomorphism. Identifying top and bottom we obtain a closed geometric braid in $S^1 \times D^2$, called the closure of the geometric braid.

S. Orevkov pointed out that for $n > 1$ the statement of Theorem 1 does not extend to the situation of two closures of quasi-positive geometric braids, one being a satellite of the other [13]. (For convenience of the reader details are given below in example 3.)

Let a tubular neighbourhood $N(K)$ of the knot K be the image of a diffeomorphism from $S^1 \times D^2$ onto $N(K)$ so that K is the image of $S^1 \times \{0\}$. (We assume that the canonical framing on $S^1 \times D^2$ is mapped to Seifert framing.) If a link L in $N(K)$ is the image of a closed geometric braid on n strands in $S^1 \times D^2$ then L is called an n -braided link around K .

For an analytic link $L \subset \partial\mathbb{B}^2$ which bounds a complex curve Y and a 4-ball $\Omega' \subset \partial\mathbb{B}^2$ whose boundary is piecewise smooth and intersects Y generically we call the intersection $Y \cap \partial\Omega'$ the Ω' -truncation of L .

The following theorem describes all links L which may appear in the situation of Theorem 1. Let as above $K \subset \partial\mathbb{B}^2$ be an analytic knot, $K = \tilde{X} \cap \partial\mathbb{B}^2$ for a smooth relatively closed complex curve \tilde{X} in a neighbourhood of \mathbb{B}^2 , and let $L \subset \partial\mathbb{B}^2$ be an analytic link, $L = \tilde{Y} \cap \partial\mathbb{B}^2$ for a relatively closed complex curve \tilde{Y} in a neighbourhood of \mathbb{B}^2 . Denote by \mathbb{D} the unit disc in the complex plane.

Theorem 2. *Let K be a smoothly analytic knot in $\partial\mathbb{B}^2$. There exists a tubular neighbourhood $N(K) \subset \partial\mathbb{B}^2$ of K and a pseudoconvex ball $\Omega' \subset \mathbb{B}^2$ with piecewise smooth boundary which is obtained from \mathbb{B}^2 by replacing a tubular neighbourhood of K in $\partial\mathbb{B}^2$ by a Levi-flat hypersurface, such that for any analytic link $L \subset N(K)$ the following holds.*

1. **(Truncation.)** *The Ω' -truncation $K' \subset \partial\Omega'$ of K is a smooth knot of the same 4-ball genus $g_4(K') = g_4(K)$ as K . If $n = w_{N(K)}(L) > 0$ then the Ω' -truncation $L' \subset \partial\Omega'$ of L is an n -braided link around K' . If $n = w_{N(K)}(L) = 0$, then L' is the empty set. The statements are true for Ω' replaced by a smoothly bounded strictly pseudoconvex domain Ω_1 , $\Omega' \subset \Omega_1 \subset \mathbb{B}^2$, (depending on K and L), with C^2 boundary which is (away from corners of $\partial\Omega'$) C^2 close to $\partial\Omega'$.*

2. **(Patterns of analytic closed braids in $N(K_1)$.)** For a strictly pseudoconvex domain Ω_1 as in statement 1 the pattern \mathcal{L}_1 of the Ω_1 -truncation L_1 of L corresponds to the conjugacy class in the braid group \mathcal{B}_n of the product of a quasi-positive braid $w \in \mathcal{B}_n$ with $g = g_4(K)$ commutators in \mathcal{B}_n .

The statements are sharp in the following sense.

3. **(Realization of patterns as analytic links.)** Let \mathcal{L} be a pattern as described in statement 2. Then for each analytic knot there exists an isotopic analytic knot $K \subset \partial\mathbb{B}^2$ such that the following holds. For any a priori given tubular neighbourhood of K , the pattern \mathcal{L} can be realized by an analytic link contained in this neighbourhood.

Notice that there is a piecewise smooth homeomorphism between $\partial\Omega'$ and $\partial\mathbb{B}^2$, and $\partial\Omega_1$ is diffeomorphic to $\partial\mathbb{B}^2$. The pattern of links $L' \subset \partial\Omega'$, and of links $L_1 \subset \partial\Omega_1$ respectively, contained in tubular neighbourhoods of knots K' , and K_1 respectively, are defined using the piecewise smooth homeomorphism, and the smooth homeomorphism respectively.

There is a continuous decreasing family $\Omega_t, t \in [0, 1]$, of strictly pseudoconvex balls Ω_t and a continuous family of contactomorphisms $\psi_t : \partial\Omega_t \rightarrow \partial\mathbb{B}^2$ such that $\Omega_0 = \mathbb{B}^2$, $\psi_0 = \text{id}$ and $\psi_t(K) = K_t \stackrel{\text{def}}{=} X \cap \partial\Omega_t$. So, one can “identify” the link L_1 of statement 2 with a closed n -braid in a tubular neighbourhood of K in $\partial\mathbb{B}^2$ (for short, with an n -braided link around K in $\partial\mathbb{B}^2$).

The pattern \mathcal{L} in statement 2 is not necessarily quasi-positive but all quasi-positive patterns can be realized in statement 3. In the quasipositive case we have the following precise statement on the 4-ball genus of the satellite.

Lemma 1. *Let $K \subset \partial\mathbb{B}^2$ be an analytic knot. Suppose L is a knot contained in a tubular neighbourhood $N(K)$ of K whose pattern \mathcal{L} is the closure of a quasi-positive n -braid. Then*

$$g_4(L) = n g_4(K) + g_4(\mathcal{L}).$$

The results of this paper grew out of an unsuccessful attempt to answer the question below. This question concerns the case of strictly pseudoconvex domains instead of the ball \mathbb{B}^2 , and is related to the following fact ([7]).

Let Ω be a strictly pseudoconvex domain in a two-dimensional Stein manifold. Then each element e of the fundamental group of the boundary $\pi_1(\partial\Omega)$ whose representatives are contractible in Ω can be represented by the boundary of an immersed analytic disc in Ω .

For an immersed analytic disc with simple transverse self-intersections the self-intersection number is the number of double points. The number of self-intersections of a general analytic disc is the number of self-intersections of its small generic perturbations.

Question. 1. What is the minimal self-intersection number among analytic discs whose boundary represents e ?

2. In particular, let $\Omega = \{(x, y, z) \in \mathbb{C}^3 : x^2 + y^3 + z^5 = \varepsilon\} \cap \mathbb{B}^3$ be the natural Stein filling of the Poincaré sphere. ($\varepsilon > 0$ is a small positive number, \mathbb{B}^3 is the unit ball in \mathbb{C}^3 .) The loops $\{x = \varepsilon^{\frac{1}{2}}\} \cap \partial\Omega$, $\{y = \varepsilon^{\frac{1}{3}}\} \cap \partial\Omega$, $\{z = \varepsilon^{\frac{1}{5}}\} \cap \partial\Omega$ bound analytic discs in Ω . Do these discs minimize the self-intersection number among analytic discs whose boundaries represent the respective element of $\pi_1(\partial\Omega)$?

Proposition 1 below implies the following. Consider the very restrictive class of analytic discs, whose boundaries are contained in a small tubular neighbourhood of one of the loops in part 2 of the question and are homotopic in $\partial\Omega$ to the respective loop. Among them the mentioned discs have the minimizing property of the self-intersection number. We do not know how to get rid of the very restrictive condition.

Notice that the question concerns the complex structure of Ω rather than its Stein homotopy type.

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2 Examples

The first two examples show that the complex curve bounded by the link L can be more complicated than the complex curve bounded by the Ω' -truncation L' which occurs in part 1 of Theorem 2. The third example shows that the statement of Theorem 1 does not extend to closures of quasi-positive geometric braids L that are 2-braided links around an analytic knot K . The example is due to S. Orevkov [13].

Example 1. (Twisted Whitehead doubles (winding number zero).)

Consider the analytic discs $\{z_2 = 0\} \cap \mathbb{B}^2$ and $\{z_1 = 0\} \cap \mathbb{B}^2$. They intersect at the origin. The union of their boundaries forms the Hopf link. Apply an automorphism of the closed ball which maps the Hopf link to an analytic link L which is the union of two circles and is contained in a 3-ball which is a subset of a small tubular neighbourhood $N(K)$ of a given analytic knot K . Join a point p on one circle by a Legendrian arc in $N(K)$ with a point q on the other circle. Choose the arc without self-intersections and without intersection points with the circles other than the endpoints. Moreover, the Legendrian arc is chosen to be the longer part of a loop representing a generator of the fundamental group of $N(K)$. Consider a partition of the Legendrian arc into small closed arcs with pairwise disjoint interior. For each small arc we take an analytic disc on \mathbb{B}^2

(which extends to a complex curve in a neighbourhood of the closed ball) such that its boundary lies on the sphere and passes through the two endpoints of the arc and through no other point of the large Legendrian arc. See figure 1. Take an analytic function f in a neighbourhood of \mathbb{B}^2 whose zero set intersects the ball \mathbb{B}^2 along the union of all these discs with the complex curve bounded by the link L . For a suitable small complex number ε the set $\{f = \varepsilon\} \cap \mathbb{B}^2$ is a smooth complex curve with connected boundary. Part of the boundary approximates a compact subset of $L \setminus (\{p\} \cup \{q\})$, the other part consists of two arcs close to the Legendrian arc. The two arcs are traveled "in opposite direction." This follows from lemmas 3.6. and 3.7 in [14].

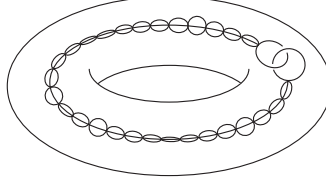


Figure 1.

Example 2. (Sum of two analytic links.)

Take two analytic links L_1 and L_2 in the tubular neighbourhood $N(K)$ of an analytic knot K . Join a point p in one of the links L_1 with a point q in the other link L_2 by a Legendrian arc in $N(K)$. The Legendrian arc is chosen without self-intersections and with interior disjoint from the two links. As in example 1 we find a complex curve X in \mathbb{B}^2 which "approximates" the union of the complex curves bounded by the links and the Legendrian arc. See figure 2. The boundary ∂X is connected. Part of it approximates a compact subset of $(L_1 \cup L_2) \setminus (\{p\} \cup \{q\})$, the other part consists of two arcs close to the Legendrian arc, the two arcs traveled in opposite direction. One of the links can be taken to be an analytic n -braided link around K , the second link may be obtained from an arbitrary analytic knot in $\partial\mathbb{B}^2$ by an automorphism of \mathbb{B}^2 which maps the second link to a 3-ball contained in $N(K)$.

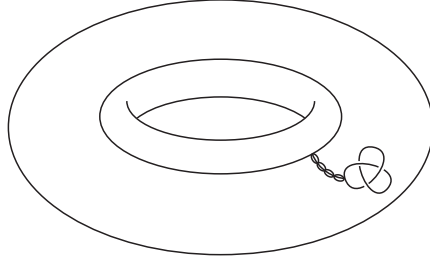


Figure 2.

Example 3. (Closures of quasi-positive geometric braids that are 2-cables of closures of quasi-positive geometric braids, see [13], corollary 2.15.)

An n -braided link in a tubular neighbourhood of a knot K is an n -cable if it is isotopic in the tubular neighbourhood to an n -braided link contained in the boundary of the tubular neighbourhood. Consider the following braid $\Delta_n^2 \cdot \sigma_{n-1} \cdot \dots \cdot \sigma_1$ in the braid group \mathcal{B}_n . Here $\sigma_1, \dots, \sigma_{n-1}$ are the standard generators and Δ_n is Garside's half-twist. (By induction $\Delta_0 = \Delta_1 = 1$, $\Delta_n = \sigma_1 \cdot \dots \cdot \sigma_{n-1} \cdot \Delta_{n-1}$.) The braid is positive (i.e., a word containing only generators, not their inverses), hence quasi-positive. The free isotopy class of its closure is represented by a smoothly analytic knot $K_1 \subset \partial \mathbb{B}^2$ ([16]). (The closure of Δ_n^2 is a link with n connected components. Hence, the closure of the considered braid is connected.) The (smooth) complex curve X_1 bounded by K_1 is (after adjusting near $\partial \mathbb{B}^2$) a branched holomorphic covering of the disc with number of branch points b_1 equal to the exponent sum of the braid (i.e., it is equal to the sum of exponents of generators of \mathcal{B}_n appearing in a representing word, see [3] or statement 2 of proposition 4 below). Hence, b_1 is equal to $n \cdot (n-1) + n-1 = n^2 - 1$. The genus $g_4(X_1)$ equals $\frac{b_1 - n + 1}{2} = \frac{n^2}{2} + O(n)$ by the Riemann-Hurwitz relation.

Orevkov proved [13], Corollary 2.15, that for large n and $N \leq \frac{8}{3}n^2 + O(n)$ the braid $\sigma_1^{-N} \Delta_{2n}^2 \in \mathcal{B}_{2n}$ is quasi-positive. Its closure is a 2-cable of Δ_n^2 . A modification of this example provides a quasi-positive braid in \mathcal{B}_{2n} whose closure is a 2-cable of K_1 and is connected. Indeed, consider $\sigma_1^{-N} \cdot c(\sigma_{n-1}) \cdot \dots \cdot c(\sigma_1) \cdot \Delta_{2n}^2$ for odd $N = \frac{8}{3}n^2 + O(n)$. Here $c(\sigma_j) \stackrel{\text{def}}{=} \sigma_{2j} \cdot \sigma_{2j-1} \cdot \sigma_{2j+1} \cdot \sigma_{2j}$, $j = 1, \dots, n-1$. See figure 3 for $n = 3$. Denote the closure of this braid by K_2 . K_2 bounds a quasi-positive surface (see [3] or statement 2 of proposition 4 below) with number of branch points b_2 equal to the exponent sum of the braid, i.e., equal to $(2n)^2 - \frac{8}{3}n^2 + O(n) = \frac{4}{3}n^2 + O(n)$, and hence of genus $\frac{2}{3}n^2 + O(n)$. Hence, for large n the lower bound for the 4-ball genus is different from that in the case of analytic satellites in small neighbourhoods of smoothly analytic knots.

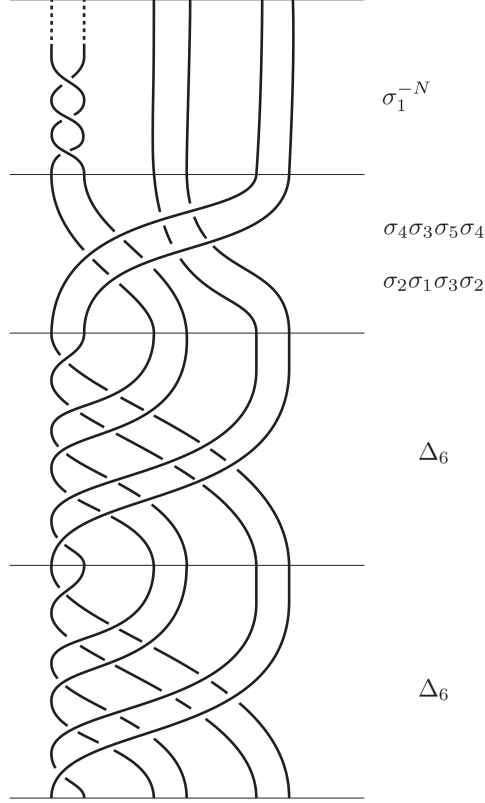


Figure 3.

3 Proof of the 4-ball genus estimates

Theorem 1 can be formulated in the more general situation when the ball \mathbb{B}^2 is replaced by a relatively compact strictly pseudoconvex domain Ω in a Stein surface $\tilde{\Omega}$ (for short, when Ω is a Stein domain). In this section we will work in this more general setting.

We start with the following simple but useful lemma.

Lemma 2. (Tubular neighbourhood of knots and of complex curves)

Let Ω be a Stein domain on a Stein surface $\tilde{\Omega}$ and let \tilde{X} be a relatively closed complex curve in $\tilde{\Omega}$ (maybe, singular) which intersects $\partial\Omega$ transversely along a knot K . Let $\tilde{X} \subset \tilde{\Omega}$ be the zero set $\tilde{X} = \{f = 0\}$ of an analytic function f in $\tilde{\Omega}$. For a relatively closed complex curve \tilde{Y} on $\tilde{\Omega}$ intersecting $\partial\Omega$ transversely along a link L , the inclusion $L \subset \{|f| < a\} \cap \partial\Omega$ for some positive number a implies the inclusion $Y = \tilde{Y} \cap \Omega \subset \{|f| < a\} \cap \Omega$.

Proof of Lemma 2. The inclusion $Y \subset \{|f| < a\} \cap \Omega$ follows from the maximum principle applied to $f|_Y$ and the fact that $\partial Y \subset \{|f| < a\} \cap \partial\Omega$. \square

Suppose the gradient of f does not vanish on $\tilde{X} \cap \bar{\Omega}$. Then, possibly after taking for $\tilde{\Omega}$ a smaller Stein surface with $\Omega \Subset \tilde{\Omega}$ and choosing a small enough number $a > 0$, the gradient of f does not vanish in a neighbourhood of the subset $\{|f| \leq a\}$ of $\tilde{\Omega}$ and there exists a holomorphic vector field $V^f = (V_1^f, V_2^f)$ near this set such that $\frac{\partial}{\partial z_1} f \cdot V_1^f + \frac{\partial}{\partial z_2} f \cdot V_2^f = a$ (see e.g. [6], Theorem 7.2.9). For a domain $\mathcal{X} \Subset \tilde{X}$ with $\tilde{X} \cap \bar{\Omega} \subset \mathcal{X}$ and for small enough a the flow of V^f defines a biholomorphic mapping ϕ^f from $\mathcal{X} \times \mathbb{D}$ onto a tubular neighbourhood $\mathcal{T}_a(\mathcal{X})$ of \mathcal{X} . We obtain a trivial holomorphic fiber bundle $\mathcal{T}_a(\mathcal{X}) \rightarrow \mathcal{X}$ with fiber $\phi^f(\{p\} \times \mathbb{D})$ over the point $p \in \mathcal{X}$. We will always consider the tubular neighbourhood $\mathcal{T}_a(\mathcal{X})$ as total space of this bundle.

Let $a > 0$ be small. Then there is a closed curve $K' \subset \mathcal{X} \cap \Omega$ such that $K \cup -K'$ bounds an annulus on \mathcal{X} and for the domain $X' \subset \mathcal{X}$ bounded by K' the tubular neighbourhood $\mathcal{T}_a(\bar{X}')$ of the closure \bar{X}' is contained in Ω . Here $-K'$ is obtained from K' by inverting orientation. The set $\mathcal{H} = \mathcal{T}_a(\partial X')$ is a Levi-flat hypersurface which is foliated into holomorphic discs $\Delta_z = \phi^f(\{z\} \times \mathbb{D})$, $z \in \partial X'$. It divides $\mathcal{T}_a(\mathcal{X})$ into two connected components. Denote by A the connected component which intersects $\partial\Omega$. For any a priori given neighbourhood of K in \mathbb{C}^2 the positive number a can be taken so small that the domain X' can be chosen so that the set A is contained in the neighbourhood of K .

Proposition 1. (Truncation and closed braids) *Let in the described situation L be a link in $N(K) = \partial\Omega \cap \{|f| < a\}$ with winding number $w_{N(K)}(L) = n$. Suppose L bounds a complex curve $Y \subset \Omega$ and \mathcal{H} intersects Y generically. Then the following holds.*

Either $n = 0$ and then $Y \cap \mathcal{H} = \emptyset$, or $n > 0$. In the latter case $L' \stackrel{\text{def}}{=} Y \cap \mathcal{H}$ is a closed n -braid in the solid torus \mathcal{H} around K' .

Proof of Proposition 1. Since by Lemma 2 Y does not meet the boundary $\bigcup_{z \in K'} \partial\Delta_z$ of \mathcal{H} the intersection number of Y with Δ_z is constant for $z \in K'$.

For a neighbourhood \tilde{X}' of \bar{X}' with $\mathcal{T}_a(\tilde{X}') \subset \Omega$ the canonical projection $\text{pr} : \mathcal{T}_a(\tilde{X}') \rightarrow \tilde{X}'$ defines a (branched) holomorphic covering $\text{pr}|_Y : Y \cap \mathcal{T}_a(\tilde{X}') \rightarrow \tilde{X}'$.

Since \mathcal{H} and Y intersect generically the latter covering is unramified in a neighbourhood of the intersection $\mathcal{H} \cap Y$. Orient K' as boundary of X' and orient L' as boundary of $Y \setminus \bar{A}$. Since a disc Δ_z is either contained in A or in ∂A or does not meet \bar{A} , the bundle projection pr maps $\mathcal{T}_a(\tilde{X}' \cap A)$ into $\tilde{X}' \cap A$ and $\mathcal{T}_a(\tilde{X}') \setminus \bar{A}$ into $\tilde{X}' \setminus \bar{A}$. Hence $\text{pr}|_Y : Y \cap \mathcal{T}_a(\tilde{X}')$ maps the side $Y \setminus \bar{A}$ of L' on Y to the side $X \setminus \bar{A}$ of K' on X , in other words $\text{pr}|_{L'} : L' \rightarrow K'$ is orientation preserving if L' and K' are oriented as boundaries of complex curves in $\mathcal{T}_a(\mathcal{X}) \setminus \bar{A}$.

The intersection number of Y with each disc Δ_z , $z \in \bar{X}'$, equals $w_{N(K)}(L) = n$. This follows from the fact that $L \cup -L'$ bounds the set $Y \cap A$ (after a small

perturbation of Y we may assume that this set is a smooth manifold). Indeed, for a neighbourhood of \bar{A} which is diffeomorphic to $K \times b^3$ let $e^{2\pi i s}$, $s \in [0, 1]$, be the parameter in the direction of K . Integrate the form ds along L and along L' and apply Stokes' theorem. \square

The 4-ball genus estimates of Theorem 1 follow from the Riemann-Hurwitz relation. Let X and Y be smooth oriented surfaces or smooth oriented surfaces with boundary. A smooth mapping $p : Y \rightarrow X$ is called a branched covering (opposed to holomorphic branched covering) if it is an orientation preserving topological covering outside the critical points, and has the form $\zeta \rightarrow \zeta^n$ for a natural number $n \geq 2$ in suitable orientation preserving complex coordinates on Y in a neighbourhood of each critical point. In case X and Y have non-empty boundary no critical point is allowed to be on the boundary of Y . The Riemann-Hurwitz relation for branched coverings of open Riemann surfaces is the following.

Let X and Y be connected open Riemann surfaces with smooth boundaries and let $p : Y \rightarrow X$ be an n -fold branched covering. Then

$$\chi(Y) = n \cdot \chi(X) - B. \quad (5)$$

Here χ is the Euler characteristic and B is the number of branch points (counted with multiplicity).

Denote by $k(X)$ the number of boundary components of a Riemann surface X . Then

$$\chi(X) = 2 - 2g(X) - k(X). \quad (6)$$

The following proposition is a direct consequence of the Riemann-Hurwitz relation and is used for the proof of Theorem 1.

Proposition 2. *Let X be a connected open Riemann surface with smooth connected boundary and let Y be an open Riemann surface. Let $p : Y \rightarrow X$ be an orientation preserving n -fold branched covering. Let Y be a Riemann surface which contains Y . Then*

$$g(Y) \geq g(Y) \geq ng(X) - (n - 1) \quad (7)$$

The second inequality is an equality if and only if Y is connected, the covering is unramified on Y and Y has n boundary components.

If Y has connected boundary then

$$g(Y) \geq g(Y) \geq ng(X) - \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (8)$$

If n is odd then $g(Y) = ng(X) - \frac{n-1}{2}$ if and only if Y and its boundary ∂Y are connected, $g(Y) = g(Y)$ and the covering $Y \rightarrow X$ is unramified.

If n is even then the equality $g(Y) = ng(X) - \left\lfloor \frac{n-1}{2} \right\rfloor$ can hold only if the boundary ∂Y has one or two components.

Proof. Let $m = m(\mathcal{Y})$ be the number of connected components of \mathcal{Y} . Denote by $\mathcal{Y}_1, \dots, \mathcal{Y}_m$ the connected components of \mathcal{Y} . Let k_j be the number of boundary components of \mathcal{Y}_j , let n_j be the multiplicity of the covering $p|_{\mathcal{Y}_j}$ and let B_j be the number of branch points (counted with multiplicity) of the latter covering. Then \mathcal{Y} has $k(\mathcal{Y}) = \sum_{j=1}^{m(\mathcal{Y})} k_j \geq 1$ boundary components, it has $B(\mathcal{Y}) = \sum_{j=1}^{m(\mathcal{Y})} B_j \geq 0$ branch points, and has covering multiplicity $\sum_{j=1}^{m(\mathcal{Y})} n_j = n$. Use for each j the relation (see (5))

$$\chi(\mathcal{Y}_j) = n_j(1 - 2g(\mathcal{X})) - B_j.$$

Consider the sum over all j . We obtain

$$\chi(\mathcal{Y}) = n(1 - 2g(\mathcal{X})) - B(\mathcal{Y}). \quad (9)$$

Apply (6) to $\mathbf{X} = \mathcal{X}$, and to \mathbf{Y} being a connected component \mathcal{Y}_j of \mathcal{Y} with the number of boundary components being k_j . Take the sum over all $m(\mathcal{Y})$ connected components of \mathcal{Y} . We obtain

$$\chi(\mathcal{Y}) = 2m(\mathcal{Y}) - 2g(\mathcal{Y}) - k(\mathcal{Y}). \quad (10)$$

Hence, since $k(\mathcal{Y}) \leq n$ and $m(\mathcal{Y}) \geq 1$ we obtain from (9) and (10)

$$2g(\mathcal{Y}) = n(2g(\mathcal{X}) - 1) + B(\mathcal{Y}) + 2m(\mathcal{Y}) - k(\mathcal{Y}) \geq n \cdot 2g(\mathcal{X}) + 2(1 - n), \quad (11)$$

and, hence, (7). The second inequality in (7) is an equality if and only if $B(\mathcal{Y}) = 0$, $m(\mathcal{Y}) = 1$, and $k(\mathcal{Y}) = n$.

Suppose Y has connected boundary (hence, Y is connected itself). Then $Y \setminus \mathcal{Y}$ is connected and has $k(\mathcal{Y}) + 1$ boundary components. Since $\chi(Y) = \chi(\mathcal{Y}) + \chi(Y \setminus \mathcal{Y})$ we obtain

$$\begin{aligned} 1 - 2g(Y) &= \chi(Y) = \chi(\mathcal{Y}) + \chi(Y \setminus \mathcal{Y}) = \\ &= n(1 - 2g(\mathcal{X})) - B(\mathcal{Y}) + 2 - 2g(Y \setminus \mathcal{Y}) - k(\mathcal{Y}) - 1. \end{aligned} \quad (12)$$

Hence,

$$g(Y) \geq ng(\mathcal{X}) - \frac{n}{2} + \frac{B(\mathcal{Y})}{2} + \frac{k(\mathcal{Y})}{2}. \quad (13)$$

Since the right hand side of equation (13) is an integral number there is at least one branch point if $n - k(\mathcal{Y})$ is odd. Hence

$$g(Y) \geq ng(\mathcal{X}) - \left\lceil \frac{n - k(\mathcal{Y})}{2} \right\rceil. \quad (14)$$

Since $k(\mathcal{Y}) \geq 1$ we obtain (8).

It is clear (see (12)) that for odd n equality $g(Y) = ng(\mathcal{X}) - \frac{n-1}{2}$ is attained if and only if $k(\mathcal{Y}) = 1$ (i.e. $\partial\mathcal{Y}$ is connected), the covering is unramified (i.e. $B(\mathcal{Y}) = 0$) and $g(Y) = g(\mathcal{Y})$. If n is even the equality in (3) can be attained only if $k(\mathcal{Y}) = 1$ or $k(\mathcal{Y}) = 2$ (i.e. if $\partial\mathcal{Y}$ has one or two connected components). The proposition is proved. \square

Proof of statements 1 and 2 of Theorem 1 for smoothly analytic knots K . Proposition 1 implies immediately that $n \geq 0$.

Let $K = \tilde{\mathcal{X}} \cap \Omega$ for a smooth relatively closed complex curve \tilde{X} in $\tilde{\Omega}$, and let for a small number $a > 0$ all values ζ with $|\zeta| \leq a$ be regular values for the defining function f of \tilde{X} . Let a be small so that for a relatively compact open subset X' of $X = \tilde{X} \cap \Omega$ with X' diffeomorphic to X the inclusion $\mathcal{T}_a(\overline{X'}) \subset \mathbb{B}^2$ holds.

Let Y be the connected complex curve bounded by L . Then $g_4(L) = g(Y)$. Indeed, let Y_s be a smooth surface in the ball \mathbb{B}^2 with the same boundary $\partial Y_s = \partial Y$ as Y . If Y_s is not connected we replace Y_s by a connected surface contained in the ball of the same genus with the same boundary. This can be done by repeating the following procedure: take a pair of connected components of Y_s , cut off a disc from each of them, and glue back an annulus contained in \mathbb{B}^2 whose interior does not meet Y_s . Assume from the beginning that Y_s is connected.

We may assume after a small perturbation which does not change the genus that $Y = \{F = 0\} \cap \mathbb{B}^2$ for a polynomial F in \mathbb{C}^2 so that Y extends to a smooth algebraic curve C in projective space \mathbb{P}^2 . We repeat now the proof of Corollary 1.3 of [11]. Let D be a smooth curve of degree six in \mathbb{P}^2 that meets C transversally and is disjoint from the ball. The double branched cover of \mathbb{P}^2 with branch locus D is a $K3$ surface. Since C and D intersect transversally the inverse image \hat{C} of C under the covering map is again a smooth complex curve.

Replace the lift of Y to the $K3$ surface by the lift of Y_s to the $K3$ surface. We get a smooth connected surface C_s in the $K3$ surface in the same homology class as the curve \hat{C} . By the theorem on the Thom conjecture the genus estimate

$$g(\hat{C}) \leq g(C_s)$$

holds. Equivalently, for the Euler characteristics we get the estimate

$$\chi(\hat{C}) \geq \chi(C_s).$$

Since the two surfaces coincide outside the lift of the ball \mathbb{B}^2 , we have

$$\chi(Y_s) \leq \chi(Y). \quad (15)$$

Since both, Y and Y_s , are connected, the inequality (15) is equivalent to the inequality

$$g(Y_s) \geq g(Y). \quad (16)$$

Inequality (16) shows that $g_4(L) = g(Y)$.

Apply Proposition 2 to Y , to X' instead of \mathcal{X} , and to $Y \cap \mathcal{T}_a(X')$ instead of \mathcal{Y} . (16) and (8) give statement 1 of Theorem 1 for smoothly analytic knots K , (16) and (7) imply statement 2 of Theorem 1 for smoothly analytic knots K . The case when K bounds a singular complex curve will be treated in the next section. \square

4 Reduction to the case of n -braided links and holomorphic coverings

Here we will prepare the proof of Theorem 2 and the proof of the remaining statements of Theorem 1. For simplicity we assume that Ω and $\tilde{\Omega}$ are strictly pseudoconvex balls in \mathbb{C}^n .

For the proof of Theorem 2 we will replace the Levi-flat hypersurface \mathcal{H} by a Levi-flat hypersurface which divides Ω rather than $\Omega \cap \{|f| < a\}$. The following lemma describes the choice.

Lemma 3. (Deformation of tubular neighbourhoods of curves on strictly pseudoconvex boundaries to Levi-flat hypersurfaces) *Let \tilde{X} be a relatively closed complex curve in $\tilde{\Omega}$ (maybe, singular) which intersects $\partial\Omega$ transversely along a knot K . There are arbitrarily small tubular neighbourhoods $N_\partial \subset \partial\Omega$ of K with the following properties. The boundary $T = \partial N_\partial$ bounds a real-analytic Levi-flat hypersurface $N_\Omega \subset \Omega$ which intersects \tilde{X} transversely along a simple closed curve K' . Moreover, the projection $N_\Omega \rightarrow K'$ defines a smooth fiber bundle with fibers being analytic discs Δ_z , $z \in K'$, with boundary $\partial\Delta_z$ in $T \subset \partial\Omega$. The union $N_\partial \cup T \cup N_\Omega$ bounds an open subset A of Ω which is diffeomorphic to $S^1 \times b^3$ (b^3 a real 3-dimensional ball), and the pseudoconvex ball $\Omega' = \Omega \setminus \bar{A}$ is diffeomorphic to Ω . Moreover, $K \cup (-K')$ bounds an annulus $\tilde{X} \cap A$ on \tilde{X} .*

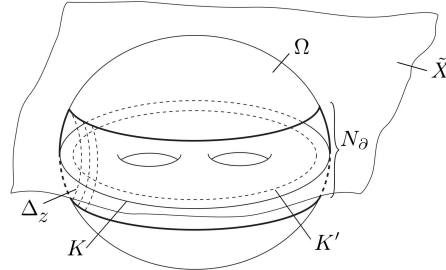


Figure 4.

Such tubular neighbourhoods N_∂ will be called good. Given any neighbourhood of K in \mathbb{C}^2 , it contains a good neighbourhood $N_\partial \subset \partial\Omega$ of K such that the respective set \bar{A} is contained in this neighbourhood.

Proof of Lemma 3. Near each boundary point the domain Ω is strictly convex in suitable holomorphic coordinates, hence there exists a smooth fiber bundle over the curve K whose fiber over each $z \in K$ is a holomorphic disc embedded into $\tilde{\Omega}$ which is complex tangent to $\partial\Omega$ at the point z and does not meet $\partial\Omega$ otherwise. Moreover, we may assume that the union of the discs forms a smooth Levi-flat hypersurface such that a neighbourhood of K on $\partial\Omega$ can be considered

as the graph over this surface of a function with non-degenerate quadratic form in the directions of the holomorphic discs. By this fact we obtain a trivialization of the smooth bundle over K from Seifert framing on a tubular neighbourhood of K on $\partial\Omega$.

Note that the holomorphic discs are transversal to \tilde{X} . Consider a C^1 approximation of the trivialized smooth bundle over K by a trivialized holomorphic disc bundle $\mathcal{T}_{ct}(\mathcal{V}) \rightarrow \mathcal{V}$ over a neighbourhood \mathcal{V} of K on \tilde{X} . We assume that \mathcal{V} is conformally equivalent to an annulus. If \mathcal{V} is small enough the holomorphic disc through each $z \in \mathcal{V}' \stackrel{\text{def}}{=} \mathcal{V} \cap \Omega$ intersects Ω along a connected simply connected set Δ_z . Choose a real analytic oriented loop K' in $\mathcal{V}' = \mathcal{V} \cap \Omega$ such that $K \cup (-K')$ bounds an annulus on \tilde{X} . Put $N_\Omega = \bigcup_{z \in K'} \Delta_z$, $T = \partial N_\Omega \subset \partial\Omega$ and let N_∂ be the connected component of $\partial\Omega \setminus T$ containing K . \square

The following variant of Proposition 1 holds.

Proposition 1'. *Let Ω , $\tilde{\Omega}$, \tilde{X} and K be as in Lemma 3. Let N_∂ be a good neighbourhood of K and let Ω' be the domain $\Omega \setminus A$ for the set A associated to N_∂ in Lemma 3. Then for any link L contained in N_∂ which equals the transverse intersection of $\partial\Omega$ with a relatively closed complex curve \tilde{Y} in $\tilde{\Omega}$ and has winding number $w_{N_\partial}(L) = n$ the following holds.*

Either $n = 0$ and then $\tilde{Y} \cap \partial\Omega' = \emptyset$, or $n > 0$. In the latter case (for a generic choice of K' and, hence a generic choice of Ω') the link $L' = \tilde{Y} \cap \partial\Omega'$ is an n -braided link (in $N_\Omega \subset \partial\Omega'$) around $K' = \tilde{X} \cap \partial\Omega'$.

The statement also holds with Ω' replaced by a strictly pseudoconvex domain Ω_1 (depending on K and L), $\Omega' \subset \Omega_1 \subset \Omega$, with C^2 boundary which is (away from corners of $\partial\Omega'$) C^2 close to $\partial\Omega'$.

The proof is a slight variation of the proof of Proposition 1 and is left to the reader.

As a corollary we obtain a proof of statements 1 and 2 of Theorem 1 for $n = 1$ in the case when K bounds a singular complex curve. Indeed, if L is contained in a good neighbourhood N_∂ of K and $w_{N_\partial}(L) = 1$ then L' is isotopic to K' in $N_\Omega \subset \partial\Omega'$, hence $g(Y \cap \Omega') = g_4(L') = g_4(K') = g_4(K)$ and $g_4(L) = g(Y) \geq g(Y \cap \Omega')$.

Suppose for some $a > 0$ the gradient of f does not vanish on the subset $\{|f| < a\}$ of $\tilde{\Omega}$, and the set $\{|f| < a\} \cap \partial\Omega$ is contained in the good neighbourhood $N_\partial \subset \partial\Omega$ of K which appears in Lemma 3.

Let Ω' and K' be as in Proposition 1'. Put $X' = \tilde{X} \cap \Omega'$. Let X'' be a relatively compact open subset of X' such that $X' \setminus \overline{X''}$ is an annulus contained in the set \mathcal{V}' of the proof of Lemma 3. For the proof of Theorem 2 we will need a smooth bundle $\mathcal{T}(\overline{X'}) \rightarrow \overline{X'}$ whose fibers are holomorphic discs. Moreover, the fibers over points in a neighbourhood of K' are contained in the discs Δ_z of the bundle $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$, and the total space of the bundle contains $\{|f| < a\} \cap \overline{\Omega'}$.

For each relatively closed complex curve \tilde{Y} in $\tilde{\Omega}$ for which $\overline{Y'} \stackrel{\text{def}}{=} \tilde{Y} \cap \overline{\Omega'}$ is contained in $\{|f| < a\}$ the intersection number of $\overline{Y'}$ with the discs of a bundle with the listed properties is constant. We will denote by $p = p_{\overline{Y'}}$ the restriction of the bundle projection to $\overline{Y'}$.

Proposition 3. (Deformation of disc bundles) *There exists a smooth disc bundle $\mathcal{T}(\overline{X'}) \rightarrow \overline{X'}$ over $\overline{X'}$, which coincides over X'' with the bundle $\mathcal{T}_{a''}(X'') \rightarrow X''$ for a small number $a'' > 0$, and coincides over a small neighbourhood of K' with a holomorphic disc bundle whose fibers are subsets of the discs of the bundle $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$. The fibers of the bundle $\mathcal{T}(\overline{X'}) \rightarrow \overline{X'}$ are holomorphic discs and the total space of the bundle contains $\{|f| < a\} \cap \overline{\Omega'}$ for a small number $a > 0$.*

Moreover, for each relatively closed complex curve \tilde{Y} in $\tilde{\Omega}$ for which $Y = \tilde{Y} \cap \Omega$ is contained in $\{|f| < a\}$, the induced mapping $p_{\overline{Y'}}$ is a branched covering with branch locus outside a neighbourhood of K' .

Further, for a suitable strictly pseudoconvex domain Ω_1 , as in Proposition 1' (which depends on \tilde{X} and \tilde{Y}) the bundle $\mathcal{T}(\overline{X'}) \rightarrow \overline{X'}$ can be extended to a smooth disc bundle $\mathcal{T}(\overline{X_1}) \rightarrow \overline{X_1}$ over $\overline{X_1} = \tilde{X} \cap \overline{\Omega_1}$ such that the union of the fibers over $K_1 = \tilde{X} \cap \partial\Omega_1$ constitute a neighbourhood of K_1 on $\partial\Omega_1$ and the projection $\overline{Y_1} \setminus Y' = \tilde{Y} \cap (\overline{\Omega_1} \setminus \Omega') \rightarrow \overline{X_1} \setminus X'$ induced by the bundle projection is an unbranched orientation preserving covering. The total space $\mathcal{T}(\overline{X_1})$ of the bundle is contained in $\overline{\Omega_1}$ and contains $\mathcal{T}_a(\tilde{X}) \cap \overline{\Omega_1}$.

We choose the trivialization of the bundle $\mathcal{T}(\overline{X_1}) \rightarrow \overline{X_1}$ so that it induces Seifert framing on $\mathcal{T}(K_1)$ for the knot $K_1 = \tilde{X} \cap \partial\Omega_1$ in $\partial\Omega_1$.

Proof of Proposition 3. Take a small number $a' > 0$. For $z \in \mathcal{V}'$ the tangent spaces at the point z to the discs of both bundles $\mathcal{T}_{a'}(\mathcal{V}') \rightarrow \mathcal{V}'$ and $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$ are transversal to \mathcal{V}' . Hence, perhaps after shrinking the holomorphic discs and using the previous notation for the bundles, we may parametrize the discs so that the following holds. The disc of the bundle $\mathcal{T}_{a'}(\mathcal{V}') \rightarrow \mathcal{V}'$ through $z \in \mathcal{V}'$ is parametrized by the natural parametrization $\varphi_z^0(\zeta)$, $\zeta \in \mathbb{D}$, as integral curve of the vector field V^f . We parametrize the disc of the bundle $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$ through $z \in \mathcal{V}'$ by $\varphi_z^1(\zeta)$, $\zeta \in \mathbb{D}$, so that for $j = 1, 2, \dots$, $\varphi_z^j(\zeta) = z + (v_z^j + g_z^j(\zeta))\zeta$, where v_z^j are non-trivial vectors in \mathbb{C}^2 which depend holomorphically on z and $g_z^j(\zeta)$ are holomorphic in z and ζ , and for all $z \in \mathcal{V}'$ the Euclidean scalar product (v_z^0, v_z^1) satisfies the inequality $|(v_z^0, v_z^1)| \geq c_0$ for a positive constant c_0 . Moreover, $|g_z^j| \leq \frac{1}{2}c_0$. The parametrization induces the previous trivialization of both bundles. Indeed, the trivialization of the bundle $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$ is related to Seifert framing on a neighbourhood of K on $\partial\Omega$ (see the proof of Lemma 3). On the other hand the trivialisation of $\mathcal{T}_{a'}(\mathcal{X})$ is given by the normal vector field on \mathcal{X} that points in the direction of positive values of the function f and the set $\{f > 0\} \cap \partial\Omega$ is, after a small generic perturbation which fixes the part of the set which is contained in a neighbourhood of K , a Seifert surface for K .

Take a small positive number ε . For each complex parameter $t \in (1 + \varepsilon)\mathbb{D}$ the mapping $G^t(z, \zeta) = t\varphi_z^0(\zeta) + (1 - t)\varphi_z^1(\zeta)$, $z \in \mathcal{V}'$, $\zeta \in \mathbb{D}$, is a holomorphic diffeomorphism of $\mathcal{V}' \times \mathbb{D}$ onto a tubular neighbourhood of \mathcal{V}' . The mappings G^t and their inverses $(G^t)^{-1}$ are uniformly bounded in the C^1 norm for $t \in (1 + \varepsilon)\mathbb{D}$.

Take a smooth function $t(z)$, $z \in \overline{X'} \setminus X'' \subset \mathcal{V}'$, with values in $(1 + \varepsilon)\mathbb{D}$, such that $t(z) = 0$ in a neighbourhood of K' and $t(z) = 1$ in a neighbourhood of $\partial X''$. Then for some small positive number δ the holomorphic discs $t(z)\varphi_z^0(\zeta) + (1-t(z))\varphi_z^1(\zeta)$, $\zeta \in \delta\mathbb{D}$, are the fibers of a smooth disc bundle over $\overline{X'} \setminus X''$ with total space contained in $\overline{\Omega'}$. Over a small neighbourhood of $\partial X''$ it coincides with the bundle $\mathcal{T}_{a''}(X') \rightarrow X'$ with $a'' = \delta a'$. Over a neighbourhood of K' the fibers are subsets of the fibers of the bundle $\mathcal{T}_{ct}(\mathcal{V}') \rightarrow \mathcal{V}'$. Extend the bundle to X'' by the holomorphic disc bundle $\mathcal{T}_{a''}(X'') \rightarrow X''$. We obtain a smooth fiber bundle $\mathcal{T}(\overline{X'}) \rightarrow \overline{X'}$ whose fibers are holomorphic discs. Moreover, the total space of the obtained bundle covers $\{|f| < a\} \cap \overline{\Omega'}$ for a small positive number a .

Let \tilde{Y} be a relatively closed complex curve in $\tilde{\Omega}$ for which $Y = \tilde{Y} \cap \Omega$ is contained in $\{|f| < a\}$. Consider a transverse intersection point of $\overline{Y'}$ with a disc of the bundle. Since the intersection is positive the induced projection $p_{\overline{Y'}}$ is an orientation preserving diffeomorphism in a neighbourhood of the intersection point. The points in a neighbourhood of $\partial Y'$ are transverse intersection points. Also, in a neighbourhood of $\overline{X''}$ the bundle is holomorphic, hence $p_{\overline{Y'}}$ is a branched holomorphic covering there.

Suppose z_0 is not in a neighbourhood of K' or in a neighbourhood of $\overline{X''}$ and $\overline{Y'}$ intersects the leaf which passes through the point z_0 of order $n \geq 2$. In coordinates (z, ζ) determined by the diffeomorphism $G^{t(z_0)}$ the disc through z_0 equals the level set $\{z = z_0\}$ and the equation of Y' in a neighbourhood of the intersection point is $\{z - z_0 = a_n(\zeta - \zeta_0)^n + \text{higher order terms}\}$.

For (z, t) close to $(z_0, t(z_0))$ the intersection of the disc $\varphi_z^t(\zeta) = t\varphi_z^0(\zeta) + (1-t)\varphi_z^1(\zeta)$, $\zeta \in \delta\mathbb{D}$, with Y' is described as follows. We have

$$\varphi_z^t(\zeta) = \varphi_z^{t(z_0)}(\zeta) + (t - t(z_0))(\varphi_z^0(\zeta) - \varphi_z^1(\zeta)), \quad \zeta \in \delta\mathbb{D}. \quad (17)$$

The second term of the sum on the right equals $(t - t(z_0))(\varphi_z^0(\zeta) - \varphi_z^1(\zeta)) = (t - t(z_0))\zeta g_z(\zeta)$, where $g_z(\zeta)$ is a holomorphic function in z and ζ such that $|g_z(\zeta)| < c$ for a constant c not depending on the parameters. Apply the inverse $(G^{t(z_0)})^{-1}$ to (17). In the obtained coordinates (z, ζ) the disc (17) is given by

$$\zeta \rightarrow (z + (t - t(z_0))\zeta h_1(z, \zeta, t - t(z_0)), \zeta + (t - t(z_0))\zeta h_2(z, \zeta, t - t(z_0))), \quad \zeta \in \delta\mathbb{D}, \quad (18)$$

where $h_j, j = 1, 2, \dots$ are bounded holomorphic functions in (z, ζ, \tilde{t}) with z in a neighbourhood of $\overline{X'} \setminus X''$, $\zeta \in \delta\mathbb{D}$, \tilde{t} in a neighbourhood of zero. Reparametrizing we may assume that h_2 is identically zero. For the intersection of Y' with the disc corresponding to φ_z^t we obtain the equation

$$z - z_0 + (t - t(z_0))\zeta h_1(z, \zeta, t - t(z_0)) = a_n(\zeta - \zeta_0)^n + \text{higher order terms}. \quad (19)$$

Let $t = t(z)$ for the chosen function $t(z)$ and suppose δ is small and z is close to z_0 . Then for $|\zeta| < \delta$ the point on the left of (19) has distance at most $\tilde{\delta}|z - z_0|$ from $z - z_0$ for a small constant $\tilde{\delta}$. Hence for each $\zeta \in \delta\mathbb{D}$ the function $z \rightarrow (z - z_0 + (t - t(z_0))\zeta h_1(z, \zeta, t - t(z_0)))^{\frac{1}{n}}$ is well-defined on the n -fold branched cover of a neighbourhood of z_0 with branch locus z_0 , and ζ can be obtained as

a quasiconformal function of $(z - z_0)^{\frac{1}{n}}$. We proved that the induced mapping $p_{\overline{Y}'}$ on \overline{Y}' is a branched covering.

The last assertion of the proposition is clear. \square

The following lemma is needed for the proofs of statement 4 of Theorem 1 and of statement 3 of Theorem 2. It provides an isotopy of an arbitrary smoothly analytic knot to a smoothly analytic knot with more convenient properties.

Lemma 4. (Isotopy to closed geometric braids) *Let K be a smoothly analytic knot. For small enough positive numbers ϵ there exists an isotopy of K to a smoothly analytic knot $K' = \mathcal{X} \cap \partial\mathbb{B}^2$, where \mathcal{X} is a relatively closed curve in $(1 + \epsilon)\mathbb{D} \times \epsilon\mathbb{D}$ such that $\partial\mathcal{X}$ is a subset of $(1 + \epsilon)\partial\mathbb{D} \times \epsilon\mathbb{D}$ and $\mathcal{X} \cap \mathbb{B}^2$ is diffeomorphic to \mathcal{X} .*

The lemma is a consequence of the following facts. Following [3] a quasipositive surface in $\mathbb{D} \times \mathbb{C}$ is a smooth proper embedding ι of a Riemann surface into $\mathbb{D} \times \mathbb{C}$ with the following property. For the canonical projection $P_1 : \mathbb{D} \times \mathbb{C} \rightarrow \mathbb{D}$ the composition $P_1 \circ \iota$ is a branched covering.

Rudolph [16] proved the following statement. *A link L in $\partial\mathbb{D} \times \mathbb{C}$ is the closure of a quasi-positive geometric braid iff it bounds a quasipositive surface in $\partial\mathbb{D} \times \mathbb{C}$. This happens iff L is isotopic in $\partial\mathbb{D} \times \mathbb{C}$ to the boundary of a complex curve.*

Bennequin [1] proved that *any oriented link in $\partial\mathbb{B}^2$ which is positively transverse to the complex tangent lines of $\partial\mathbb{B}^2$ is isotopic through links with the same property (for short, it is transverse isotopic) to a closed braid (in $\partial\mathbb{B}^2 \setminus \{z_1 = 0\}$).*

Sketch of proof of Lemma 4. Let X be the complex curve in the ball bounded by K . Put $K_1 = K$ and let K_t be a transverse isotopy in $\partial\mathbb{B}^2$ to a knot K_2 which is contained in $\mathbb{C} \times \epsilon\mathbb{D}$ for a small positive number ϵ and is the closure of a braid. Consider the set

$$X' = X \cup \bigcup_{t \in [1, 2]} tK_t \cup \bigcup_{t \in [2, \infty]} tK_2.$$

Here $tK_t \stackrel{\text{def}}{=} \{tz : z \in K_t\}$. Boileau and Orevkov [3] proved that, after smoothing, the set $X' \cap (2\mathbb{D} \times \mathbb{C})$ is (diffeomorphic to) a quasi-positive surface in $2\mathbb{D} \times \mathbb{C}$. By Rudolph's theorem the link $L_0 \stackrel{\text{def}}{=} X' \cap (2\partial\mathbb{D} \times \mathbb{C})$ is isotopic in $2\partial\mathbb{D} \times \mathbb{C}$ to the boundary $L_1 \stackrel{\text{def}}{=} \partial\mathcal{X}'$ of a complex curve \mathcal{X}' . We may assume that \mathcal{X}' is a subset of a quasipositive surface in $(2 + 2\epsilon)\mathbb{D} \times \mathbb{C}$ such that the projection to the first factor has branch locus in $2\mathbb{D}$. By contraction in the z_2 -direction the isotopy can be chosen so that $L_1 = \partial\mathcal{X}'$ (and hence also the complex curve \mathcal{X}') is contained in $2\overline{\mathbb{D}} \times 2\epsilon\mathbb{D}$. The isotopy provides a family of links L_t , $t \in [0, 1]$, in $2\partial\mathbb{D} \times \mathbb{C}$.

Put $K' = \frac{1}{2}(\mathcal{X}' \cap 2\partial\mathbb{B}^2)$. It remains to prove that K_2 is isotopic to K' (since K is isotopic to K_2). Note that for a positive constant β we have the inclusion

$K_2 \subset \{|z_1| > \beta\}$. Consider the part $2\partial\mathbb{B}^2 \cap \{|z_1| > 2\beta\}$ of the sphere of radius 2. Assign to each point z in this set the point of intersection of the ray $\{tz : t > 1\}$ with the set $\{|z_1| = 2\} \times \mathbb{C}$. We obtain a diffeomorphism F of $2\partial\mathbb{B}^2 \cap \{|z_1| > 2\beta\}$ onto a subset of $\{|z_1| = 2\} \times \mathbb{C}$. Consider the inverse $F^{-1}(L_t)$, $t \in [0, 1]$, of Rudolph's isotopy. This is an isotopy of links in $2\partial\mathbb{B}^2$ with $F^{-1}(L_0) = 2K_2$. If ϵ is small enough then $F^{-1}(L_1) = F^{-1}(\partial\mathcal{X}')$ approximates $\mathcal{X}' \cap 2\partial\mathbb{B}^2$ well enough and, hence, is isotopic to it. The lemma is proved. \square

Let X be a smooth 2-manifold (or a smooth 2-manifold with boundary). We call an embedding of a smooth 2-manifold (or of a smooth 2-manifold with boundary) Y into the product $X \times \mathbb{D}$ an n -horizontal embedding if for the natural projection $P_X : X \times \mathbb{D} \rightarrow X$ the restriction $P_X|_Y : Y \rightarrow X$ is an n -covering (smooth and unramified).

Proposition 4. *Let ϵ be a small positive number. Suppose an open Riemann surface \mathcal{X} with smooth connected boundary is holomorphically embedded into $(1 + \epsilon)\mathbb{D} \times \epsilon\mathbb{D}$ in such a way that the mapping $P_1|_{\mathcal{X}} : \mathcal{X} \rightarrow (1 + \epsilon)\mathbb{D}$ is a simple branched covering with branch locus in \mathbb{D} . Suppose $i : \bar{\mathcal{Y}} \rightarrow \bar{\mathcal{X}} \times \mathbb{D}$ is a smooth n -horizontal embedding of the closure of an open Riemann surface \mathcal{Y} into $\bar{\mathcal{X}} \times \mathbb{D}$. Then the following statements hold.*

1. (Isotopy of n -horizontal embeddings to holomorphic embeddings)

There exists a simply connected smoothly bounded domain $\mathcal{D} \subset \mathbb{D}$ containing the branch locus of $P_1|_{\mathcal{X}}$ such that the following holds. Put $X = (P_1|_{\mathcal{X}})^{-1}(\mathcal{D}) = \mathcal{X} \cap (\mathcal{D} \times \mathbb{C})$ and $Y = \mathcal{Y} \cap (X \times \mathbb{D})$. The embedding of the two-manifold Y into $X \times \mathbb{D}$ is isotopic through horizontal embeddings to a holomorphic embedding of a Riemann surface into $X \times \mathbb{D}$.

2. (Isotopy classes of boundary links)

Denote by $w_Y \in \mathcal{B}_n$ a braid whose conjugacy class corresponds to the isotopy class of the boundary link $\partial\mathcal{Y} \subset \partial\mathcal{X} \times \mathbb{D}$ of statement 1. Let $w \in \mathcal{B}_n$ be a quasi-positive braid. Then with \mathcal{D} as in statement 1 there exists a domain \mathcal{D}_1 , $\mathcal{D} \subset \mathcal{D}_1 \subset \mathbb{D}$, such that with $X_1 = (P_1)^{-1}(\mathcal{D}_1)$ there exists a smooth embedding $i : \bar{Y}_1 \rightarrow \bar{X}_1 \times \mathbb{D}$ of the closure \bar{Y}_1 of an open Riemann surface Y_1 which is holomorphic on the Riemann surface Y_1 and such that $P_X|_{Y_1}$ is a branched n -covering of X_1 and the isotopy class of the boundary link $\partial Y_1 \subset \partial X_1 \times \mathbb{D}$ corresponds to the conjugacy class of $w \cdot w_Y$. The number of branch points equals the exponent sum of the braid w .

Notice that the complex curves X and \mathcal{X} are diffeomorphic. By a contraction in the z_2 -direction we may also assume that \mathcal{X} and $\mathcal{X} \cap \mathbb{B}^2$ are diffeomorphic.

Postpone the proof of Proposition 4.

The following lemma relates the boundaries of the embedded surfaces in Proposition 4 to analytic links which are braided links around analytic knots.

Lemma 5. (Boundary links of horizontally embedded surfaces and braided links around knots) *Let $\epsilon > 0$ be a sufficiently small number, and let*

\mathcal{X} be a Riemann surface which is holomorphically embedded into $(1+\epsilon)\mathbb{D} \times \epsilon\mathbb{D}$ so that $P_1|_{\mathcal{X}} : \mathcal{X} \rightarrow (1+\epsilon)\mathbb{D}$ is a branched covering with branch locus contained in \mathbb{D} such that \mathcal{X} is diffeomorphic to $\mathcal{X} \cap \mathbb{B}^2$. Denote by K the knot $K = \mathcal{X} \cap \partial\mathbb{B}^2$.

Let $\mathcal{D} \subset \mathbb{D}$ be a simply connected smoothly bounded domain containing the branch locus of $P_1|_{\mathcal{X}}$ so that $X = \mathcal{X} \cap (\mathcal{D} \times \mathbb{C})$ is diffeomorphic to \mathcal{X} . Suppose there is an open Riemann surface Y and a holomorphic embedding $i : Y \rightarrow \mathcal{T}_a(X)$ into a small tubular neighbourhood of X such that for the projection $P_X : \mathcal{T}_a(X) \rightarrow X$ the mapping $p = P_X \circ i$ is a branched holomorphic n -covering.

Then there exists an isotopy of K in $\partial\mathbb{B}^2$ to a smoothly analytic knot \tilde{K} such that any a priori given tubular neighbourhood of \tilde{K} contains an analytic link \tilde{L} which is an n -braided link around \tilde{K} with pattern equal to the isotopy class of ∂Y in $\mathcal{T}_a(\partial X)$. The complex curve \tilde{Y} in \mathbb{B}^2 bounded by \tilde{L} is diffeomorphic to Y .

Proof of Lemma 5. Let \mathcal{D}_t , $t = [0, 1]$, be a continuous decreasing family of simply connected smoothly bounded domains with $\mathcal{D}_0 = \mathbb{D}$ and $\mathcal{D}_1 = \mathcal{D}$. Let α be a continuous function on $[0, 1]$ with $\alpha(0) = 1$ and $\alpha(t) > 1$ for $t > 0$. Denote by $\omega_t : \mathcal{D}_t \rightarrow \alpha(t)\mathbb{D}$ the conformal mapping with $\omega_t'(0) > 0$. The ω_t depend continuously on t .

Let $t \rightarrow s(t)$, $t \in [0, 1]$, be a continuous decreasing positive function with $s(0) = 1$ which takes sufficiently small values for t away from 0. For $z = (z_1, z_2) \in \mathcal{D}_t \times \mathbb{C}$, $t \in [0, 1]$, we put $\mathcal{G}_t(z) = (\omega_t(z_1), s(t) \cdot z_2)$. The mapping $\mathcal{G}_t|_{\mathcal{X} \cap (\mathcal{D}_t \times \mathbb{C})}$ is a conformal map onto a (relatively closed) complex curve X_t in $\alpha(t)\mathbb{D} \times \epsilon\mathbb{D}$. If ϵ is small enough then for suitable choices of the functions α and s the intersection of each X_t with $\partial\mathbb{B}^2$ is transversal, and, hence, $K_t = X_t \cap \partial\mathbb{B}^2$, $t \in [0, 1]$, is a transversal isotopy. Moreover, if $s(1) > 0$ is small then the subset $\tilde{X} = X_1 \cap \mathbb{B}^2$ of X_1 is close to X_1 and is diffeomorphic to X_1 . Put $\tilde{K} = \partial\tilde{X}$.

We may assume that $a > 0$ is sufficiently small and Y is identified with an embedded submanifold of $\mathcal{T}_a(X)$. The Riemann surface $Y_1 = \mathcal{G}_1(Y)$ is conformally equivalent to Y and embedded into the tubular neighbourhood $\mathcal{G}_1(\mathcal{T}_a(X))$ of X_1 . For a suitable choice of the function α and small a the Riemann surface $\tilde{Y} = Y_1 \cap \mathbb{B}^2$ is diffeomorphic to Y_1 and the boundary $\tilde{L} \stackrel{\text{def}}{=} \partial\tilde{Y} \subset \partial\mathbb{B}^2$ is an n -braided link around $\partial\tilde{X}$ contained in the small tubular neighbourhood $\mathcal{G}_1(\mathcal{T}_a(X)) \cap \partial\mathbb{B}^2$ of \tilde{K} . Moreover, the isotopy class of $\partial\tilde{Y}$ in $\mathcal{G}_1(\mathcal{T}_a(X))$ equals the pattern of \tilde{L} . \square

For the proof of Proposition 4 it will be convenient to work with the following terminology.

A continuous mapping from a smooth manifold (or a smooth manifold with boundary) X into the set of all monic polynomials $\overline{\mathfrak{P}}_n$ of degree n is a quasipolynomial of degree n . It can be written as function in two variables $x \in X$, $\zeta \in \mathbb{C}$, i.e. $\mathcal{P}(x, \zeta) = a_0(x) + a_1(x)\zeta + \dots + a_{n-1}(x)\zeta^{n-1} + \zeta^n$, for continuous functions a_j , $j = 1, \dots, n$, on X . If the image of the map is contained in the space \mathfrak{P}_n of monic polynomials of degree n without multiple zeros, it is called separable. For a separable quasipolynomial, considered as a function \mathcal{P}

on $X \times \mathbb{C}$, its zero set $\mathfrak{S}_{\mathcal{P}} = \{(x, \zeta) \in X \times \mathbb{C}, \mathcal{P}(x, \zeta) = 0\}$ is a surface which is n -horizontally embedded into $X \times \mathbb{C}^n$. Vice versa, each n -horizontally embedded 2-manifold in $X \times \mathbb{C}$ corresponds to a separable quasi-polynomial of degree n on X . An isotopy of separable quasi-polynomials (i.e. a family of separable quasi-polynomials depending continuously on a parameter in $[0, 1]$) is equivalent to an isotopy of n -horizontally embedded manifolds.

Notice that the set \mathfrak{P}_n is biholomorphic to $\mathbb{C}^n \setminus \{D_n = 0\}$, where D_n denotes the discriminant.

Proof of Proposition 4. For the **proof of assertion 1** we consider a simple smooth arc Γ (i.e. a diffeomorphic image of the closed unit interval) in the disc \mathbb{D} which passes once through each point of the branch locus E of the covering $\mathcal{X} \rightarrow (1 + \epsilon)\mathbb{D}$. Let $\Gamma_{\mathcal{X}} = \mathcal{X} \cap (P_1)^{-1}(\Gamma) \subset \mathcal{X}$ and let \mathcal{P} be the mapping from \mathcal{X} into $C_n(\mathbb{C})/\mathcal{S}_n \cong \mathfrak{P}_n \cong \mathbb{C}^n \setminus \{D_n = 0\} \subset \mathbb{C}^n$ defined by the embedding of \mathcal{Y} into $\mathcal{X} \times \mathbb{D}$. The subset $\Gamma_{\mathcal{X}}$ of \mathcal{X} has no interior point. Hence, by Mergelyan's Theorem for Riemann surfaces [10] the restriction $\mathcal{P} \mid \Gamma_{\mathcal{X}}$ can be approximated uniformly on $\Gamma_{\mathcal{X}}$ by an analytic mapping of a neighbourhood $X \subset \mathcal{X}$ of $\Gamma_{\mathcal{X}}$ into \mathbb{C}^n . After perhaps shrinking X , we may assume that the restriction $P_1 \mid X$ is a branched covering onto a domain $\mathcal{D} \subset \mathbb{D}$. The set $\Gamma_{\mathcal{X}}$ is a deformation retract of \mathcal{X} . After shrinking X we may assume that X is diffeomorphic to \mathcal{X} .

If the approximation is good enough then, after, perhaps, shrinking X again, the image of X under the approximating mapping is contained in the symmetrized configuration space. We obtain a holomorphic mapping of X into the symmetrized configuration space which is isotopic to $\mathcal{P} \mid X$ through smooth mappings into symmetrized configuration space. The isotopy of mappings defines an isotopy of n -horizontal embeddings into $X \times \mathbb{D}$ which joins the embedding of $\mathcal{Y} \cap (X \times \mathbb{D})$ with a holomorphic n -horizontal embedding of a complex curve into $X \times \mathbb{D}$. We proved assertion 1.

For the **proof of assertion 2** we need Lemma 6 below. The following construction prepares its statement. Let \mathcal{X} be an open Riemann surface with smooth connected boundary. Take any smoothly bounded domain $\mathcal{X}_0 \subset \mathcal{X}$ which is a strong deformation retract of \mathcal{X} , and take simply connected smoothly bounded domains $\mathcal{X}_j \Subset \mathcal{X} \setminus \mathcal{X}_0$, $j = 1, \dots, k$, with pairwise disjoint closure. Consider simple smooth pairwise disjoint arcs $\gamma_j : [0, 1] \rightarrow \mathcal{X}$, $j = 1, \dots, k$, such

that for each j the interior of γ_j is contained in $\mathcal{X} \setminus \bigcup_{j=0}^k \mathcal{X}_j$ and γ_j joins a boundary

point of \mathcal{X}_0 with a boundary point of \mathcal{X}_j . Consider disjoint "rectangles" R_j around the γ_j , i.e. for some $\varepsilon > 0$ the R_j are diffeomorphic mappings of $(-\varepsilon, \varepsilon) \times [0, 1]$ with image in \mathcal{X} , and such that $R_j \mid \{0\} \times [0, 1]$ equals γ_j and the sides $R_j((-\varepsilon, \varepsilon) \times \{0\})$ and $R_j((-\varepsilon, \varepsilon) \times \{1\})$ are contained in $\partial\mathcal{X}_0$, respectively

in $\partial\mathcal{X}_j$. Suppose the domain $\mathcal{X}_0 \cup \bigcup_{j=1}^k \mathcal{X}_j \cup \bigcup_{j=1}^k R_j((-\varepsilon, \varepsilon) \times [0, 1])$ has smooth boundary (see fig. 5). Then it is again a deformation retract of \mathcal{X} . We call any

domain of the above described type a thickening of $\bigcup_{j=0}^k \mathcal{X}_j \cup \bigcup_{j=1}^k \gamma_j$.

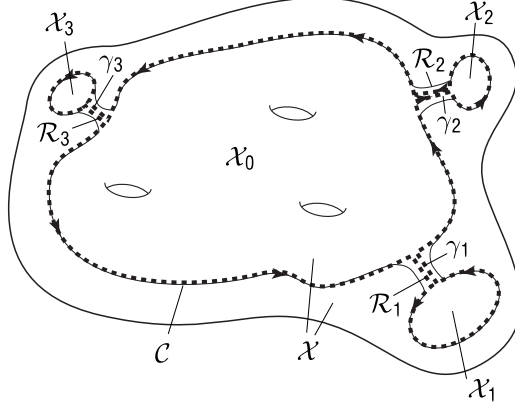


Figure 5.

Denote by $P_{\mathcal{X}} : \mathcal{X} \times \mathbb{D} \rightarrow \mathcal{X}$ the canonical projection to the first factor. Consider a branched holomorphic covering p from an open Riemann surface \mathcal{Y} onto \mathcal{X} , and a holomorphic embedding i of \mathcal{Y} into the disc bundle $\mathcal{X} \times \mathbb{D}$ such that $P_{\mathcal{X}} \circ i = p$ (for short i lifts p). The following Lemma 6 provides the embedding of another open Riemann surface Y into the holomorphic disc bundle over a deformation retract X of \mathcal{X} such that the embedding lifts a branched covering with "more" branch points and has the following property: the isotopy class of the embedding of the boundary ∂Y into $\partial X \times \mathbb{D}$ differs from that of $\partial \mathcal{Y} \subset \partial \mathcal{X} \times \mathbb{D}$ by a prescribed quasi-positive braid.

Lemma 6. (Adding branch points) *Let \mathcal{X} and \mathcal{Y} be open Riemann surfaces with smooth boundary. Suppose the boundary of \mathcal{X} is connected. Let $i : \overline{\mathcal{Y}} \rightarrow \overline{\mathcal{X}} \times \mathbb{D}$ be a smooth embedding which is holomorphic on \mathcal{Y} and such that for the canonical projection $P_{\overline{\mathcal{X}}} : \overline{\mathcal{X}} \times \mathbb{D} \rightarrow \overline{\mathcal{X}}$ the restriction $P_{\overline{\mathcal{X}}} \mid \mathcal{Y}$ is a (simple branched or unbranched) holomorphic n -covering of \mathcal{X} . Denote by $w_{\mathcal{Y}} \in \mathcal{B}_n$ a braid whose conjugacy class corresponds to the isotopy class of the boundary link $\partial \mathcal{Y} \subset \partial \mathcal{X} \times \mathbb{D}$. Let $w \in \mathcal{B}_n$ be a quasi-positive braid of exponent sum m . Let \mathcal{X}_0 be as above. Suppose $\mathcal{X} \setminus \mathcal{X}_0$ does not intersect the branch locus of $P_{\mathcal{X}} \mid \mathcal{Y}$.*

Then for $k \geq m$ and for each collection $\mathcal{X}_0, \mathcal{X}_1, \dots, \mathcal{X}_k, \gamma_1, \dots, \gamma_k$, as above there exists a thickening $X \subset \mathcal{X}$ of $\bigcup_{j=0}^k \mathcal{X}_j \cup \bigcup_{j=1}^k \gamma_j$ and a holomorphic embedding of an open Riemann surface $i : Y \rightarrow X \times \mathbb{D}$ into the disc bundle over X such that $P_{\mathcal{X}} \mid Y$ is a simple branched n -covering of X whose branch locus contains exactly one point in each member of a chosen collection of m sets among the \mathcal{X}_j , $j = 1, \dots, k$, and may contain also some points in \mathcal{X}_0 . Moreover, the isotopy class of the boundary link $\partial Y \subset \partial X \times \mathbb{D}$ corresponds to the conjugacy class of the braid $w \cdot w_{\mathcal{Y}} \in \mathcal{B}_n$. The same conclusion holds for any thickening of

$$\bigcup_{j=0}^k \mathcal{X}_j \cup \bigcup_{j=1}^k \gamma_j \text{ which is contained in } X.$$

End of Proof of assertion 2 of Proposition 4. Let \mathcal{D} be the domain chosen in assertion 1, let $d_j, j = 1, \dots, m$, be pairwise disjoint smoothly bounded open topological discs contained in $\mathbb{D} \setminus \mathcal{D}$, and let $\Gamma_j, j = 1, \dots, n$, be pairwise disjoint arcs with interior contained in $\mathbb{D} \setminus (\bar{\mathcal{D}} \cup \bigcup \bar{d}_j)$ such that Γ_j joins a boundary point of \mathcal{D} with a boundary point of d_j . Let $\mathcal{D}_1 \subset \mathbb{D}$ be a small enough thickening of $\mathcal{D} \cup \bigcup (d_j \cup \Gamma_j)$. Then $(P_1)^{-1}(\mathcal{D}_1)$ is a small thickening of $(P_1)^{-1}(\mathcal{D} \cup \bigcup (d_j \cup \Gamma_j))$. Apply Lemma 6, where we take $\mathcal{X}_0 = (P_1)^{-1}(\mathcal{D}) \subset \mathcal{X}$, $\mathcal{X}_j, j = 1, \dots, k$, running over all components of the preimages under P_1 of the d_i for all i , and $\gamma_j, j = 1, \dots, k$, (with the respective label) running over all components of the preimages of all Γ_i . Choose the collection of m of the \mathcal{X}_j 's so that P_1 is injective on the union of the sets of the collection.

Lemma 6 implies statement 2 of Proposition 4. \square

Proof of Lemma 6. Denote by \mathcal{A} the compact set $\mathcal{A} = \bigcup_{j=0}^k \bar{\mathcal{X}}_j \cup \bigcup_{j=1}^k \gamma_j$. Let \mathcal{C} be the (oriented) closed curve which is contained in the boundary $\partial \mathcal{A}$ and surrounds \mathcal{A} counterclockwise (see fig. 5). \mathcal{C} is obtained from $\partial \mathcal{X}_0$ (traveled counterclockwise) in the following way: for each $j = 1, \dots, k$, we cut $\partial \mathcal{X}_0$ at the starting point q_j of γ_j and insert the curve γ_j followed by $\partial \mathcal{X}_j$ traveled from the endpoint of γ_j surrounding \mathcal{X}_j counterclockwise, followed by $-\gamma_j$ (γ_j with inverse orientation).

Consider the closed geometric braids $\partial \mathcal{Y} \subset \partial \mathcal{X} \times \mathbb{C}$ and $\mathcal{Y} \cap (\partial \mathcal{X}_0 \times \mathbb{C})$. Let Q be a point in $\partial \mathcal{X}$ and let $q = q_1 \in \partial \mathcal{X}_0$ be the initial point of the arc γ_1 . Put $E_n^Q = P_{\mathbb{C}}(\mathcal{Y} \cap (\{Q\} \times \mathbb{C}))$ and $E_n^q = P_{\mathbb{C}}(\mathcal{Y} \cap (\{q\} \times \mathbb{C}))$. Here $P_{\mathbb{C}} : \mathcal{X} \times \mathbb{C} \rightarrow \mathbb{C}$ is the canonical projection.

Denote by Γ an arc whose interior parametrizes $\partial \mathcal{X} \setminus \{Q\}$ (with orientation induced from the orientation of $\partial \mathcal{X}$ as boundary of \mathcal{X}) and whose endpoints are equal to Q . Let Γ_0 be the respective object for $\partial \mathcal{X}_0$ and the point q . The closed geometric braid $\partial \mathcal{Y}$ defines a continuous map from Γ to \mathfrak{P}_n with base point E_n^Q , the closed geometric braid $\mathcal{Y} \cap (\partial \mathcal{X}_0 \times \mathbb{C})$ defines a continuous map from Γ_0 to \mathfrak{P}_n with base point E_n^q .

Let γ be a closed arc with interior contained in $\mathcal{X} \setminus \left(\bigcup_{j=0}^k \bar{\mathcal{X}}_j \cup \bigcup_{j=1}^k \gamma_j \right)$, with initial point q and with terminating point Q . The embedding of $\bar{\mathcal{Y}}$ into $\bar{\mathcal{X}} \times \mathbb{C}$ also defines a continuous map of $\gamma + \Gamma + (-\gamma)$ to \mathfrak{P}_n which is another geometric braid with base point E_n^q . (The sum of pathes means, that we first walk along γ until we reach its terminating point, which is the initial point of Γ , then along Γ , then along $-\gamma$, which means γ equipped with orientation opposite to the orientation of γ .) After identifying the pathes Γ_0 and $\gamma + \Gamma + (-\gamma)$ by a homeomorphism, the two geometric braids are isotopic, since $\mathcal{X} \setminus \mathcal{X}_0$ does not contain points in the branch locus of the projection $P_{\mathcal{X}} | \mathcal{Y}$. Denote the isotopy class of these braids by $w'_{\mathcal{Y}}$. Having in mind a homomorphism between \mathcal{B}_n and

the group of isotopy classes of geometric braids with base point E_n^q , we write $w'_y \stackrel{\text{def}}{=} w_1^{-1} \cdot w_y \cdot w_1$ for a braid $w_1 \in \mathcal{B}_n$.

Pick m of the \mathcal{X}_j , $j \geq 1$, and label them in the order we meet the initial point q_j of γ_j when traveling along $\partial\mathcal{X}_0$ in the direction of orientation of $\partial\mathcal{X}_0$ as boundary of \mathcal{X}_0 , starting from q . For transparency of the proof we assume $m = k$. The braid w is quasi-positive. Hence, we may write $w' \stackrel{\text{def}}{=} w_1 \cdot w \cdot (w_1)^{-1} = (v_1)^{-1} \sigma_{\ell_1} v_1 \cdot \dots \cdot (v_m)^{-1} \sigma_{\ell_m} v_m$ for $v_j \in \mathcal{B}_n$, $j = 1, \dots, m$.

Consider the n points in E_n^q and label them by ξ_1, \dots, ξ_n . Put $\varrho = \min\{|\xi_i - \xi_j| : i \neq j\}$. For each j we pick a point $\eta_j \in \mathcal{X}_j$ and consider the holomorphic curve $\mathcal{Z}_j = \{(z_1, z_2) \in \mathcal{X}_j \times \mathbb{C} : \alpha_j (z_2 - \xi_{\ell_j})^2 = z_1 - \eta_j\}$. Here α_j is a constant chosen so that \mathcal{Z}_j is contained in $\mathcal{X}_j \times \{|z_2 - \xi_{\ell_j}| < \frac{\varrho}{2}\}$. Consider the surfaces $\mathcal{Y}_j = \mathcal{Z}_j \cup \bigcup_{i \neq \ell_j, \ell_{j+1}} \{(z_1, z_2) : z_2 = \xi_i, z_1 \in \mathcal{X}_j\}$, $j = 1, \dots, m$. Each \mathcal{Y}_j is a Riemann surface which has $m-1$ connected components and smooth boundary. The boundary $\partial\mathcal{Y}_j$ corresponds to the conjugacy class of braids which contains σ_{ℓ_j} . The projection $P_{\mathcal{X}}|_{\mathcal{Y}_j} : \mathcal{Y}_j \rightarrow \mathcal{X}_j$ is a simple branched covering with a single point η_j in the branch locus. The embedding of the closure $\overline{\mathcal{Y}_j}$ into $\mathcal{X}_j \times \mathbb{C}$ corresponds to a quasi-polynomial, denoted $\mathcal{P}_j(z, \zeta)$, $z \in \mathcal{X}_j, \zeta \in \mathbb{C}$, such that the polynomial $\zeta \rightarrow \mathcal{P}_j(\eta_j, \zeta)$, $\zeta \in \mathbb{C}$, has a double zero, and the restriction of the quasi-polynomial to $\mathcal{X}_j \setminus \{\eta_j\}$ is separable.

Let Q_1 be the terminating point of γ_1 . Denote by Γ_1 a closed arc whose interior parametrizes $\partial\mathcal{X}_1 \setminus \{Q_1\}$ with orientation induced from the orientation of $\partial\mathcal{X}_1$ and whose endpoints are equal to Q_1 . Define a mapping from $\overline{\mathcal{X}_0} \cup \gamma_1 \cup \overline{\mathcal{X}_1}$ into the space \mathfrak{P}_n of all monic polynomials of degree n as follows. For $z_1 \in \overline{\mathcal{X}_0}$ we let the map be equal to the monic polynomial with zeros $\overline{\mathcal{Y}} \cap (\{z_1\} \times \mathbb{C})$, and for $z_1 \in \overline{\mathcal{X}_1}$ we let the map be equal to the monic polynomial with zeros $\overline{\mathcal{Y}_1} \cap (\{z_1\} \times \mathbb{C})$. We extend the map continuously by a map from γ_1 to the space \mathfrak{P}_n of polynomials without multiple roots so that the following holds. The induced mapping from $\gamma_1 + \Gamma_1 + (-\gamma_1) + \Gamma_0$ to \mathfrak{P}_n with base point E_n^q represents the geometric braid $(v_1)^{-1} \sigma_{\ell_1} v_1 \cdot w'_y$. (We use the same homomorphism between \mathcal{B}_n and the group of isotopy classes of geometric braids with base point E_n^q as before.)

Continue by induction. Let Q_2 be the terminating point of γ_2 . Denote by Γ_2 an arc whose interior parametrizes $\partial\mathcal{X}_2 \setminus \{Q_2\}$ with orientation induced from the orientation of $\partial\mathcal{X}_2$ and whose endpoints are equal to Q_2 . Denote by Γ_0^1 the path obtained by walking along $\partial\mathcal{X}_0$ according to its orientation from $q = q_1$ until the initial point q_2 of the curve γ_2 . Consider the continuous map from $\overline{\mathcal{X}_0} \cup \gamma_1 \cup \overline{\mathcal{X}_1} \cup \gamma_2 \cup \overline{\mathcal{X}_2}$ into the space \mathfrak{P}_n which is equal to the previous map on $\overline{\mathcal{X}_0} \cup \gamma_1 \cup \overline{\mathcal{X}_1}$, which is equal to the monic polynomial with zeros $\overline{\mathcal{Y}_2} \cap (\{z_2\} \times \mathbb{C})$ for $z_1 \in \overline{\mathcal{X}_2}$, and which is continuously extended to γ_2 so that the following holds. The induced mapping from $(\gamma_1 + \Gamma_1 + (-\gamma_1)) + (\Gamma_0^1 + \gamma_2 + \Gamma_2 + (-\gamma_2) + (-\Gamma_0^1)) + \Gamma_0$ has image in \mathfrak{P}_n and represents the braid $(v_1)^{-1} \sigma_{\ell_1} v_1 \cdot (v_2)^{-1} \sigma_{\ell_2} v_2 \cdot w'_y$.

By induction we obtain a continuous map from $\mathcal{A} = \overline{\mathcal{X}_0} \cup \bigcup_{j=1, \dots, m} (\gamma_j \cup \overline{\mathcal{X}_j})$ to $\overline{\mathfrak{P}_n}$ which is holomorphic on $\bigcup_{j=0, \dots, m} \mathcal{X}_j$. Moreover, the induced map on the curve \mathcal{C} defines a geometric braid corresponding to the conjugacy class of $(v_1)^{-1} \sigma_{\ell_1} v_1 \cdot \dots \cdot (v_m)^{-1} \sigma_{\ell_m} v_m \cdot w'_y = w_1 w w_1^{-1} w_1 w_y w_1^{-1} = w_1 w w_y w_1^{-1}$.

The mapping from \mathcal{A} to $\overline{\mathfrak{P}}_n$ corresponds to a quasi-polynomial of degree n on \mathcal{A} denoted by \mathcal{P} , $\mathcal{P}(z, \zeta) = \sum_{m=0}^n a_m(z) \zeta^m$, $z \in \mathcal{A}$, $\zeta \in \mathbb{D}$. Here $a_n \equiv 1$. The coefficients a_m are continuous functions on \mathcal{A} which are analytic on the interior $\text{Int } \mathcal{A} = \bigcup_{j=0}^k \mathcal{X}_j$. The polynomial $\zeta \rightarrow \mathcal{P}(z, \zeta)$ has multiple zeros exactly if z is in the branch locus of $P_{\mathcal{X}} | \mathcal{Y}$. This happens for some points in \mathcal{X}_0 and for the points $\eta_j \in \mathcal{X}_j$, $j = 1, \dots, m$. Denote by \mathcal{Y}' the zero set of the quasi-polynomial in $\mathcal{A} \times \mathbb{C}$.

By the Mergelyan theorem for open Riemann surfaces ([10], Corollary 4) each coefficient a_m , $m = 0, 1, \dots, n-1$, can be uniformly approximated by a holomorphic function \tilde{a}_m in a neighbourhood $\tilde{\mathcal{A}}$ of \mathcal{A} on \mathcal{X} . Denote $\tilde{\mathcal{P}}(z, \zeta) = \sum_{m=0}^n \tilde{a}_m(z) \zeta^m$, $z \in \tilde{\mathcal{A}}$, $\zeta \in \mathbb{D}$. Here $\tilde{a}_n = a_n \equiv 1$. We may assume that 0 is a regular value of $\tilde{\mathcal{P}}$. If the approximation is good enough then for each $z \in \partial \mathcal{A}$ the polynomial $\tilde{\mathcal{P}}(z, \cdot)$ has n distinct roots which are close to the roots of $\mathcal{P}(z, \cdot)$. There is a thickening $X \subset \tilde{\mathcal{A}}$ of $\bigcup_{j=0}^k \mathcal{X}_j \cup \bigcup_{j=1}^k \gamma_j$ such that $\tilde{\mathcal{P}}(z, \cdot)$ has no multiple roots for z in the closure of each rectangle R_j added to build the thickening. Put $\bar{Y} = \{(z, \zeta) \in \bar{X} \times \mathbb{D} : \tilde{\mathcal{P}}(z, \zeta) = 0\}$. The interior Y is holomorphically embedded into $X \times \mathbb{D}$ and the projection $P_{\mathcal{X}} | \bar{Y} : \bar{Y} \rightarrow \bar{X}$ extends to a neighbourhood of \bar{Y} as branched covering with branch locus in $\bigcup_{j=0}^k \mathcal{X}_j \subset X$. In particular, if \bar{X} is close enough to \mathcal{A} , then the closed geometric braids $\partial Y \subset \partial X \times \mathbb{C}$ and $\mathcal{Y} \cap (\mathcal{C} \times \mathbb{C})$ are free isotopic (after identifying ∂X and \mathcal{C} by a homeomorphism), and, hence, they correspond to the same conjugacy class in \mathcal{B}_n , namely to the conjugacy class of $w \cdot w_{\mathcal{Y}}$. The same conclusion holds for X replaced by any thickening of $\bigcup_{j=0}^k \mathcal{X}_j \cup \bigcup_{j=1}^k \gamma_j$ contained in X . The lemma is proved. \square

Note that the proof of Lemma 6 provides also a proof of one of the implications of Rudolph's theorem. Indeed, let L be the closure of a quasipositive braid. Put $\mathcal{X} = \mathbb{D}$, let the \mathcal{X}_j be as in the lemma and consider a constant mapping from \mathcal{X}_0 to \mathfrak{P}_n . The lemma provides a simply connected domain $X \subset \mathbb{D}$ and a holomorphic embedding of a Riemann surface Y into $X \times \mathbb{C}$ such that $Y \rightarrow X$ is a simple branched covering and Y has smooth boundary ∂Y such that the embedding $\partial Y \rightarrow \partial X \times \mathbb{C}$ defines a closed geometric braid which corresponds to the conjugacy class of the quasi-positive braid. To obtain a complex curve in $\mathbb{D} \times \mathbb{C}$ with boundary contained in $\partial \mathbb{D} \times \mathbb{C}$ and isotopic to L , we map X conformally onto \mathbb{D} .

For the proofs of the remaining statements of the theorems we need the following known proposition.

Proposition 5. *Let X be a connected smooth surface or a connected smooth surface with boundary. Let $\pi_1(X, x_0)$ be the fundamental group of X with a*

given base point x_0 . The following statements hold.

1. There is a one-to-one correspondence between homomorphisms $\Psi : \pi_1(X, x_0) \rightarrow \mathcal{S}_n$ and unramified n -coverings $p : Y \rightarrow X$ with given label of points in the fiber $p^{-1}(x_0)$.
2. There is a one-to-one correspondence between homomorphisms $\Phi : \pi_1(X, x_0) \rightarrow \mathcal{B}_n$ and isotopy classes of separable quasi-polynomials with fixed value E_n at x_0 and given label of points in E_n . The quasi-polynomial lifts a covering p iff the homomorphism Φ lifts the homomorphism $p_* : \pi_1(X, x_0) \rightarrow \mathcal{S}_n$ corresponding to p , i.e. $p_* = \tau_n \circ \Phi$ for the canonical homomorphism $\tau_n : \mathcal{B}_n \rightarrow \mathcal{S}_n$.
3. The connected components of the covering space Y of an unramified holomorphic n -covering $p : Y \rightarrow X$ are in bijective correspondence to the orbits of $p_*(\pi_1(x, z_0)) \subset \mathcal{S}_n$ on the set consisting of n points. In particular, Y is connected iff $p_*(\pi_1(X, x_0))$ acts transitively.
4. Suppose X is a 2-manifold of genus g with boundary. Suppose the boundary ∂X is connected and the base point x_0 is contained in the boundary. Denote by $\{\partial X\}$ the element of the fundamental group $\pi_1(X, x_0)$ which is represented by traveling along ∂X in the direction of orientation as boundary of X starting from x_0 . (So, $\{\partial X\}$ is the product of g commutators of suitable generators of the fundamental group.) Let $p : Y \rightarrow X$ be an unramified n -covering and let $\mathcal{P}(x, \zeta)$ be a separable quasi-polynomial lifting it, i.e. the covering is equivalent to $P_X \mid \mathfrak{S}_{\mathcal{P}} : \mathfrak{S}_{\mathcal{P}} \rightarrow X$.

Then the connected components of the boundary ∂Y correspond to the orbits of the single even permutation $p_*(\{\partial X\})$. $p_*(\{\partial X\})$ is the product of g commutators in \mathcal{S}_n .

Moreover, the free isotopy class of the boundary link $\partial \mathfrak{S}_{\mathcal{P}} \subset \partial X \times \mathbb{C}$ is a closed geometric braid representing the conjugacy class of the product of g commutators in \mathcal{B}_n .

Notice that a conjugate of a commutator is again a commutator.

5 Proof of the remaining statements of the theorems

Proof of statement 3 of Theorem 1.

The sharpness of the estimate for the general case of links L (statement 3) is an immediate consequence of the following lemma.

Lemma 7. *Let \mathcal{X} be an open Riemann surface with smooth connected boundary of genus $g(\mathcal{X}) \geq 1$. For any natural $n \geq 1$ there exists an unbranched holomorphic n -covering \mathcal{Y} of \mathcal{X} such that \mathcal{Y} is connected and has n boundary*

components. Moreover, there is a holomorphic embedding of \mathcal{Y} into the disc bundle $\mathcal{X} \times \mathbb{D}$ that lifts the covering map.

Indeed, let K be a smoothly analytic knot, i.e. $K = \partial\mathbb{B}^2 \cap \mathcal{X}$ for a complex curve $\mathcal{X} = \{z \text{ in a neighbourhood of } \overline{\mathbb{B}^2} : f(z) = 0\}$. Here f is an analytic function with non-vanishing gradient in a neighbourhood of \mathcal{X} . Suppose $g(\mathcal{X}) \geq 1$. We may assume that $\mathcal{X} \cap \mathbb{B}^2$ is diffeomorphic to \mathcal{X} . Identify the disc bundle $\mathcal{X} \times \mathbb{D}$ with a small tubular neighbourhood of \mathcal{X} in \mathbb{C}^2 and consider the Riemann surface \mathcal{Y} of Lemma 7 to be an embedded submanifold of the tubular neighbourhood. For these \mathcal{X} and \mathcal{Y} equality in the Riemann-Hurwitz relation (7) is obtained, since the covering is unbranched, \mathcal{Y} is connected and the number of boundary components of \mathcal{Y} is maximal. Let $L = \mathcal{Y} \cap \partial\mathbb{B}^2$. If the tubular neighbourhood is small enough then \mathcal{Y} is diffeomorphic to $\mathcal{Y} \cap \mathbb{B}^2$. It follows that in this case equality is attained in (4). Statement 3 is proved.

Proof of Lemma 7. Let $\mathcal{F} : \mathbb{C} \rightarrow \mathbb{C}$ be the complex linear mapping $\mathcal{F}(\zeta) = e^{\frac{2\pi i}{n}} \zeta$. The n -th iterate \mathcal{F}^n is the identity. The fundamental group $\pi_1(\mathcal{X}, x_0)$ is a free group on $g = g(\mathcal{X})$ generators. Denote by $\langle \mathcal{F} \rangle$ the group of self-homeomorphisms of \mathbb{C} generated by \mathcal{F} . Let $\Phi : \pi_1(\mathcal{X}, x_0) \rightarrow \langle \mathcal{F} \rangle$ be a homomorphism which assigns the element \mathcal{F} to one of the generators of the fundamental group, and assigns to each of the other generators any element of the group $\langle \mathcal{F} \rangle$. By proposition 4 the image $\Phi(\{\partial\mathcal{X}\})$ is the identity.

Let $\tilde{\mathcal{X}}$ be the universal covering of \mathcal{X} . The fundamental group $\pi_1(\mathcal{X}, x_0)$ acts on $\tilde{\mathcal{X}} \times \mathbb{C}$ as follows.

$$\tilde{\mathcal{X}} \times \mathbb{C} \ni (x, \zeta) \rightarrow (\gamma(x), \Phi(\gamma)(\zeta)), \quad \gamma \in \pi_1(\mathcal{X}, x_0). \quad (20)$$

The action is free and properly discontinuous. Hence, the mapping

$$p : \tilde{\mathcal{X}} \times \mathbb{C} \rightarrow \mathcal{E} \stackrel{\text{def}}{=} \tilde{\mathcal{X}} \times \mathbb{C} / \pi_1(\mathcal{X}, x_0) \quad (21)$$

is a holomorphic covering map. It defines a holomorphic fiber bundle over \mathcal{X} with fiber being a complex line, and with transition functions being complex linear. Since \mathcal{X} is open this holomorphic line bundle is trivial. The covering map (21) respects a holomorphic foliation on $\tilde{\mathcal{X}} \times \mathbb{C}$, namely the trivial foliation with leaves $\tilde{\mathcal{X}} \times \{\zeta\}$, $\zeta \in \mathbb{C}$. The map \mathcal{F} permutes the points of the set $E = \{1, e^{\frac{2\pi i}{n}}, \dots, e^{\frac{2\pi i(n-1)}{n}}\}$ along a cycle of length n . Hence, the covering map (21) maps the set $\tilde{\mathcal{X}} \times E$ to a single leaf. This leaf is a Riemann surface which we denote by \mathcal{Y} and identify with a surface which is n -horizontally embedded into the trivial bundle $\mathcal{X} \times \mathbb{C}$. The projection $P_{\mathcal{X}} : \mathcal{X} \times \mathbb{C} \rightarrow \mathcal{X}$ restricts to \mathcal{Y} as unramified covering. The covering corresponds to a homomorphism $\Psi : \pi_1(\mathcal{X}, x_0) \rightarrow \langle s \rangle$, where $s = (12 \dots n)$ is a cycle of length n , and $\langle s \rangle$ is the subgroup of the symmetric group generated by s . Hence, $\Psi(\{\partial\mathcal{X}\}) = \text{id}$, and \mathcal{Y} has n boundary components by Proposition 5. \square

Proof of statement 4 of Theorem 1.

The sharpness of the bound for the case when L is also required to be a knot (statement 4) is obtained as follows. Let K be an analytic knot

with $g_4(K) = g \geq 1$. Using the isotopy provided by Lemma 4 we may assume that $K = \mathcal{X} \cap \partial\mathbb{B}^2$ for a smooth complex curve \mathcal{X} contained in $(1 + \epsilon)\mathbb{D} \times \epsilon\mathbb{D}$ for a small positive number ϵ such that $P_1 \mid \mathcal{X} : \mathcal{X} \rightarrow (1 + \epsilon)\mathbb{D}$ is a branched covering with branch locus in \mathbb{D} and \mathcal{X} is diffeomorphic to $\mathcal{X} \cap \mathbb{B}^2$.

If n is odd there exists an unbranched holomorphic covering $p : \mathcal{Y} \rightarrow \mathcal{X}$ such that \mathcal{Y} has connected boundary. This follows immediately from Proposition 5 and a theorem of Ore which says that each even permutation is a commutator. For convenience of the reader we provide the following simple examples on commutators. Example 1 together with statements 1 and 4 of Proposition 5 provide the required unbranched covering. Indeed, take the covering corresponding to the homomorphism $\Psi : \pi_1(\mathcal{X}, x_0) \rightarrow \mathcal{S}_n$ for which $\Psi(a_1) = s_1$, $\Psi(a_2) = s_2$, $\psi(a_j) = \text{id}$, $j = 3, \dots, 2g$, for a suitable choice of generators a_j of $\pi_1(\mathcal{X}, x_0)$ and for the permutations s_1 and s_2 of Example 1. Example 2 will be used below.

Example 1. Suppose n is an odd number, $n = 2m + 1$. Consider the following two permutations $s_1 = (23) \dots (2m \ 2m + 1)$ and $s_2 = (12)(34) \dots (2m - 1 \ 2m)$ in \mathcal{S}_n . Then the commutator $s = [s_1, s_2]$ is a cycle of order n . (See fig. 6a for $n = 7$.)

Example 2. Let $n = 2m$ be an even number. Consider the permutations $s_1 = (23) \dots (2m - 2 \ 2m - 1)$ and $s_2 = (12) \dots (2m - 1 \ 2m)$ in \mathcal{S}_n . The commutator $s = [s_1, s_2]$ is the disjoint union of two cycles of order $\frac{n}{2}$. Note that the subgroup $\langle s_1, s_2 \rangle$ of \mathcal{S}_n generated by s_1 and s_2 acts transitively on $\{1, 2, \dots, n\}$. (See fig. 6b for $n = 8$.)

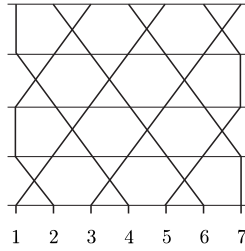


fig. 6a

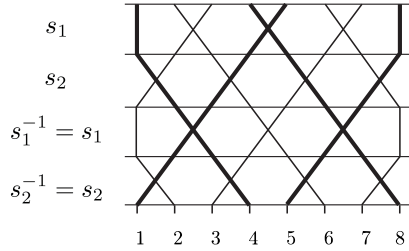


fig. 6b

Figure 6.

Take the complex structure on \mathcal{Y} which is induced by the covering map p . Consider the holomorphic mapping $i = (p, f) : \mathcal{Y} \rightarrow \mathcal{X} \times \mathbb{D}$. Here f is a bounded holomorphic function. The mapping i is an immersion. It is an embedding of $p^{-1}(\mathcal{X}_0)$ into $\mathcal{X}_0 \times \mathbb{D}$ for a domain $\mathcal{X}_0 \subset \mathcal{X}$ iff f separates the points of the fiber $p^{-1}(z)$ for each $z \in \mathcal{X}_0$. Choose the holomorphic function $f : \mathcal{Y} \rightarrow \mathbb{D}$ so that it separates all points of the fibers $p^{-1}(z_j)$ for all points z_j in the branch

locus of $P_1 \mid \mathcal{X}$. This is possible since \mathcal{X} is an open Riemann surface with smooth connected boundary, hence it has bounded holomorphic functions with prescribed values at finitely many points. From the identity theorem applied to $f \circ \varphi_j$ for the local inverses φ_j of p it follows that the function f separates points of fibers for all $z \in \mathcal{X}$ not belonging to a discrete subset Λ of \mathcal{X} . The disc \mathbb{D} contains only finitely many points of $P_1(\Lambda)$. Join them by pairwise disjoint simple smooth arcs with points in $\partial\mathbb{D}$. Remove small neighbourhoods of the arcs from \mathbb{D} so that we obtain a simply connected smoothly bounded domain $\mathcal{D} \subset \mathbb{D}$. We may assume that \mathcal{D} contains the branch locus of $P_1 \mid \mathcal{X}$, in other words, $\mathcal{X}_1 = \mathcal{X} \cap (\mathcal{D} \times \mathbb{C})$ is diffeomorphic to \mathcal{X} . Identify the disc bundle $\mathcal{X}_1 \times \mathbb{D}$ with a small enough tubular neighbourhood of \mathcal{X}_1 in \mathbb{C}^2 . We obtain an embedding of $\mathcal{Y}_1 = p^{-1}(\mathcal{X}_1)$ into the tubular neighbourhood of \mathcal{X}_1 such that \mathcal{Y}_1 is diffeomorphic to \mathcal{Y} . By the Riemann-Hurwitz relation we obtain

$$1 - 2g(\mathcal{Y}_1) = \chi(\mathcal{Y}_1) = n \cdot \chi(\mathcal{X}_1) = n(1 - 2g(\mathcal{X}_1)),$$

hence, since \mathcal{X}_1 is diffeomorphic to \mathcal{X} and \mathcal{Y}_1 is diffeomorphic to \mathcal{Y} , we obtain

$$g(\mathcal{Y}) = n \cdot g(\mathcal{X}) - \frac{n-1}{2}.$$

Lemma 5 provides a further isotopy to an analytic knot K and an analytic knot L which is an n -braided link around K contained in a small tubular neighbourhood of K such that equality in relation (3) for the 4-ball genus bound is attained.

It remains to consider the case when n is even. We may assume that \mathcal{X} is as in the case of odd n . There exists an unbranched holomorphic n -covering $p : \mathcal{Y} \rightarrow \mathcal{X}$ with \mathcal{Y} connected and with boundary consisting of two connected components. This follows from statements 1 and 4 of Proposition 5 and Example 2 above. (See fig. 6b for the case $n = 8$.) By statement 2 of Proposition 5 there is a smooth separable quasipolynomial \mathcal{P} that lifts the covering. Hence, we may identify $\mathcal{Y} = \mathfrak{S}_{\mathcal{P}}$ with the zero set of the quasipolynomial so that the covering map p equals $P_{\mathcal{X}} \mid \mathcal{Y} : \mathcal{Y} \rightarrow \mathcal{X}$. Shrinking \mathcal{X} we may assume that \mathcal{X} and \mathcal{Y} have smooth boundary and $\bar{\mathcal{Y}}$ is smoothly n -horizontally embedded into $\bar{\mathcal{X}} \times \mathbb{D}$.

Let $w_{\mathcal{Y}} \in \mathcal{B}_n$ represent the conjugacy class corresponding to the isotopy class of $\partial\mathcal{Y}$ in $\partial\mathcal{X} \times \mathbb{D}$. Take for w a conjugate of a generator of \mathcal{B}_n which permutes two strands of $w_{\mathcal{Y}}$ corresponding to different connected components of $\partial\mathcal{Y}$ (in other words, the closure of $w \cdot w_{\mathcal{Y}}$ defines a connected closed braid). Apply statement 2 of Proposition 4. We obtain a domain $X_1 \subset \mathcal{X}$ of the form $X_1 = P_1^{-1}(\mathcal{D}_1)$ for a smoothly bounded simply connected domain \mathcal{D}_1 , $\mathcal{D}_1 \subset \mathbb{D}$, such that X_1 is diffeomorphic to \mathcal{X} . Also we obtain an embedding of an open Riemann surface $i : Y_1 \rightarrow X_1 \times \mathbb{D}$ into the disc bundle such that $P_{\mathcal{X}} \mid Y_1$ is a branched holomorphic covering with a single branch point and with connected boundary $\partial Y_1 \subset \partial X_1 \times \mathbb{D}$ which determines a closed geometric braid which (after identifying $\partial\mathcal{X}$ with ∂X_1 by a homeomorphism) represents the conjugacy class of $w \cdot w_{\mathcal{Y}}$.

By the Riemann-Hurwitz relation

$$1 - 2g(Y_1) = n(1 - 2g(X_1)) - 1,$$

hence

$$g(Y_1) = ng(X_1) - \frac{n-2}{2}.$$

Since X_1 is diffeomorphic to \mathcal{X} , and, hence, to $\mathcal{X} \cap \partial\mathbb{B}^2$, we have $g_4(K) = g(X_1)$. Lemma 5 gives a further isotopy of K to a smoothly analytic knot again denoted by K and an analytic knot L in a small neighbourhood of K which is an n -braided link around K such that $g_4(L) = g(Y_1)$. The first part of statement 4 is proved.

The following example proves the last part of statement 4. Embed the standard punctured torus holomorphically into \mathbb{C}^2 using the Weierstraß \wp -function:

$$(\mathbb{C} \setminus (\mathbb{Z} + i\mathbb{Z})) / (\mathbb{Z} + i\mathbb{Z}) \ni \zeta \rightarrow (\wp(\zeta), \wp'(\zeta)) \in \mathbb{C}^2. \quad (22)$$

Denote the image of the embedding by \mathcal{X} . Let R be a large positive number. The intersection $X_R = \frac{1}{R}\mathcal{X} \cap \mathbb{B}^2$ is a torus with a hole. If R is large then X_R contains a domain \mathcal{R} which is adjacent to ∂X_R and is conformally equivalent to an annulus of conformal module larger than $\frac{\pi}{2} \frac{1}{\log \frac{3+\sqrt{5}}{2}}$. (Recall that for $0 \leq r_1 < r_2 \leq \infty$ the conformal module of the annulus $\{r_1 < |z| < r_2\}$ in the complex plane equals $\frac{1}{2\pi} \log \frac{r_2}{r_1}$.) Put $X = X_R \subset \mathbb{B}^2$ for a number R with this property. Let $K = \partial X \subset \partial\mathbb{B}^2$. Then K is a smoothly analytic knot.

Suppose for any $a > 0$ there exists an analytic knot L contained in the tubular neighbourhood $N(K) = \partial\mathbb{B}^2 \cap \mathcal{T}_a(\frac{1}{R}\mathcal{X})$, such that $n = w_{N(K)}(L) = 3$ and equality is obtained in the 4-ball genus estimate (3) for K and L . Let Y be the complex curve bounded by L . Apply Lemma 2 and Proposition 1. Let \mathcal{H} be the Levi-flat hypersurface of Proposition 1 and let A be the set defined before the statement of Proposition 1. Put $X' = X \setminus \bar{A}$, $Y' = Y \setminus \bar{A}$, $L' = Y \cap \mathcal{H}$. If $a > 0$ is small then A is contained in a small neighbourhood of K in \mathbb{C}^2 .

If R is large then the set $\mathcal{R}' \stackrel{\text{def}}{=} X' \cap \mathcal{R}$ is conformally equivalent to an annulus of conformal module close to that of \mathcal{R} , in particular the conformal module of \mathcal{R}' is larger than $\frac{\pi}{2} \frac{1}{\log \frac{3+\sqrt{5}}{2}}$.

Apply Proposition 2 with $\mathcal{X} = X'$ and $\mathcal{Y} = Y'$: Since $n = 3$ is odd, $L = \partial Y$ is connected and equality holds in (3), by Proposition 2 the Riemann surface $\mathcal{Y} = Y'$ has connected boundary and the covering is unramified. The embedding of Y' in the disc bundle over X' defines a holomorphic map of X' to \mathfrak{P}_n . Its restriction to \mathcal{R}' is a holomorphic map into \mathfrak{P}_n which represents the free isotopy class of $L' = \partial Y'$ (a commutator class by Proposition 4). Since the conformal module of \mathcal{R}' is large, by Lemma 8.3 of [8] the class of L' is the conjugacy class of a pure braid, i.e. L' cannot be connected. The contradiction proves the last part of statement 4 of Theorem 1. Theorem 1 is proved. \square

Proof of Theorem 2. Statement 1 follows from Proposition 1'.

We will prove now statement 2. Let \tilde{X} and \tilde{Y} be relatively closed complex curves in a neighbourhood $\tilde{\Omega}$ of \mathbb{B}^2 such that $K = \tilde{X} \cap \partial\mathbb{B}^2$ and $L = \tilde{Y} \cap \partial\mathbb{B}^2$. For the domain Ω_1 of Proposition 1' (with $\Omega = \mathbb{B}^2$) we let $X_1 = \tilde{X} \cap \Omega_1$, $Y_1 = \tilde{X} \cap \Omega_1$, $K_1 = \partial X_1$ and $L_1 = \partial Y_1$. Consider the bundle $\mathcal{T}(\overline{X_1}) \rightarrow \overline{X_1}$ of proposition 3

with trivialization inducing Seifert framing for K_1 on $\partial\Omega_1$. The intersection of Y_1 with the discs of the trivialized bundle defines a continuous mapping from X_1 to the space $\overline{\mathfrak{P}_n}$ of monic polynomials of degree n . The restriction of this mapping to ∂X_1 is a mapping to the space \mathfrak{P}_n of monic polynomials of degree n without multiple zeros which represents the pattern of the braided link ∂Y_1 .

If the covering $p : \bar{Y}_1 \rightarrow \bar{X}_1$ induced by the bundle projection is unramified then by Proposition 5 the pattern \mathcal{L}_1 of the link ∂Y_1 is the conjugacy class of a product of $g = g(X_1) = g_4(K)$ commutators in \mathcal{B}_n .

Consider now the general case. Let $d \subset X_1$ be a smoothly bounded simply connected domain which contains the branch locus of p such that $\bar{X}_1 \setminus d$ is diffeomorphic to \bar{X}_1 (in particular, $\partial X_1 \cap \partial d \neq \emptyset$). (For example, one can take for d the union of the following sets: suitable neighbourhoods of simple disjoint arcs joining a critical value of p with a boundary point of X_1 , and a suitable simply connected part of a collar of ∂X_1 in X_1 .) Put $\mathcal{Y}_d \stackrel{\text{def}}{=} p^{-1}(d)$ and use the following notation: $\mathcal{X}_d \stackrel{\text{def}}{=} d$, $\mathcal{X}_{Cd} \stackrel{\text{def}}{=} X_1 \setminus \bar{d}$ and $\mathcal{Y}_{Cd} \stackrel{\text{def}}{=} p^{-1}(\mathcal{X}_{Cd})$.

Take a base point q in \bar{X}_1 which is a boundary point of $X_1 \cap \partial d$ (in particular, $q \in \partial d \cap \partial X_1$). Choose the point $E_n^q = P_{\mathbb{D}}(\mathcal{Y} \cap (\{q\} \times \mathbb{D}))$ as base point in the symmetrized configuration space.

Let Γ_d be an arc whose interior parametrizes $\partial \mathcal{X}_d \setminus \{q\}$ and whose two endpoints are equal to q . Respectively, we denote by Γ_{Cd} an arc whose interior parametrizes $\partial \mathcal{X}_{Cd} \setminus \{q\}$ and whose two endpoints are equal to q . Both arcs are equipped with orientation induced by orienting $\partial \mathcal{X}_d$, or $\partial \mathcal{X}_{Cd}$, respectively, as boundaries of the domains \mathcal{X}_d , and \mathcal{X}_{Cd} , respectively.

The n -horizontal embeddings $\partial \mathcal{Y}_d \subset \partial \mathcal{X}_d \times \mathbb{C}$, and $\partial \mathcal{Y}_{Cd} \subset \partial \mathcal{X}_{Cd} \times \mathbb{C}$ respectively, define continuous mappings from Γ_d into \mathfrak{P}_n , and from Γ_{Cd} into \mathfrak{P}_n , respectively. Identifying \mathcal{B}_n with the group of isotopy classes of geometric braids with base point E_n^q we obtain (after identifying Γ_d and Γ_{Cd} with the unit interval) two braids w_d and w_{Cd} . Since d is simply connected the braid w_d is quasi-positive by Rudolph's theorem. Since the covering over \mathcal{X}_{Cd} is unramified the braid w_{Cd} is a product of g commutators in \mathcal{B}_n . (See statement 4 of Proposition 5.)

Let Γ_{X_1} be an arc whose interior parametrizes $\partial X_1 \setminus \{q\}$ (with orientation induced from the orientation of ∂X_1 as boundary of X_1) and whose two endpoints are equal to q . The isotopy class of the continuous mapping from Γ_{X_1} to \mathfrak{P}_n which is defined by the n -horizontal embedding $\partial Y_1 \subset \partial X_1 \times \mathbb{C}$ is equal (after identification of the curves $\Gamma_d + \Gamma_{Cd}$ and Γ_{X_1}) to the braid $w_d \cdot w_{Cd}$. Statement 2 is proved.

It remains to prove statement 3. By Lemma 4 after an isotopy we are in the situation when the knot is equal to $\mathcal{X} \cap \partial \mathbb{B}^2$ for a smooth relatively closed complex curve \mathcal{X} in a small neighbourhood of $\{z_2 = 0\}$ such that for a small positive number ϵ the mapping $P_1 : \mathcal{X} \rightarrow (1 + \epsilon)\mathbb{D}$ is a branched covering with branch locus in \mathbb{D} and $\mathcal{X} \cap \mathbb{B}^2$ is diffeomorphic to \mathcal{X} . We may assume that \mathcal{X} has smooth boundary.

Let $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g]$, $\alpha_j, \beta_j \in \mathcal{B}_n$ for $j = 1, \dots, g$, be a product of $g = g(\mathcal{X})$ commutators. By statements 2 and 4 of Proposition 5 there exists a smooth

n -horizontal embedding of a smooth surface with boundary $\overline{\mathcal{Y}}$, $\overline{\mathcal{Y}} \rightarrow \overline{\mathcal{X}} \times \mathbb{C}$, such that the embedding of the boundary $\partial\mathcal{Y} \rightarrow \partial\mathcal{X} \times \mathbb{C}$ corresponds to the conjugacy class of the afore mentioned product of commutators. Indeed, let $\overline{\mathcal{Y}}$ be the zero set of the quasi-polynomial which corresponds to the homomorphism Φ for which $\Phi(a_j) = \alpha_j$, $\Phi(b_j) = \beta_j$ for suitable generators a_j and b_j , $j = 1, \dots, g$, of the fundamental group.

Let $w \in \mathcal{B}_n$ be a quasipositive braid such that the pattern \mathcal{L} is the conjugacy class of the braid $w \cdot [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]$. By Proposition 4 there is an open subset X_1 of \mathcal{X} diffeomorphic to \mathcal{X} and a holomorphically embedded manifold $Y_1 \subset X_1 \times \mathbb{C}$ so that for the canonical projection $P_{\mathcal{X}}$ the mapping $P_{\mathcal{X}}|_{Y_1} : Y_1 \rightarrow X_1$ is a holomorphic branched n -covering. The number of branch points B equals the exponent sum of the braid w . Moreover, the isotopy class of the link $\partial Y_1 \subset \partial X_1 \times \mathbb{C}$ corresponds to the conjugacy class of the braid $w \cdot [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]$.

By Lemma 5 the conjugacy class \mathcal{L} of $w \cdot [\alpha_1, \beta_1] \cdot \dots \cdot [\alpha_g, \beta_g]$, can be realized by an analytic link contained in an a priori given neighbourhood of a knot $K \subset \partial\mathbb{B}^2$ which is isotopic to $\mathcal{X} \cap \partial\mathbb{B}^2$.

Theorem 2 is proved. \square

Proof of Lemma 1. The proof uses the proof of statement 3 of Theorem 2. We may assume after an isotopy which moves K and L that the knot K has the form $\mathcal{X} \cap \partial\mathbb{B}^2$ with \mathcal{X} as in the proof of statement 3. The pattern \mathcal{L} of L is the closure of a quasipositive braid w (it corresponds to the case of Theorem 2 when the product of commutators is the identity). Let B be the exponent sum of w . For Y_1 and X_1 as in the proof of statement 3 of Theorem 2 the following equation for the Euler characteristic holds

$$\chi(Y_1) = n\chi(X_1) - B. \quad (23)$$

Under the condition of Lemma 1 the closure of the quasipositive braid w is connected. Notice the following fact. For a knot L' in a tubular neighbourhood of the unknot which represents the closure of a quasipositive braid with exponent sum B we have

$$1 - 2g_4(L') = n - B. \quad (24)$$

Indeed, we may assume after an isotopy that $L' = \partial\mathbb{B}^2 \cap Y'$ for a complex curve Y' in a neighbourhood of \mathbb{B}^2 such that $P_1|_{Y'} : Y' \rightarrow (1 + \epsilon)\mathbb{D}$ is a branched n -covering with branch locus in \mathbb{D} and Y' is diffeomorphic to $Y' \cap \mathbb{B}^2$. Then $g_4(L') = g(Y')$. Hence,

$$1 - 2g_4(L') = \chi(Y') = n\chi((1 + \epsilon)\mathbb{D}) - B = n - B.$$

Equations (23) and (24) with the pattern \mathcal{L} of Lemma 1 instead of L' imply

$$1 - 2g(Y_1) = n(1 - 2g(X_1)) - B = -2ng(X_1) + (n - B) = -2ng(X_1) + 1 - 2g_4(\mathcal{L}).$$

As in the proof of statement 3 of Theorem 2 Lemma 5 provides an isotopy of $\mathcal{X} \cap \partial\mathbb{B}^2$ to a smoothly analytic knot \tilde{K} , which bounds a Riemann surface

$\tilde{X} \subset \mathbb{B}^2$ that is diffeomorphic to \mathcal{X} and thus to X_1 , and an n -braided link \tilde{L} in a small tubular neighbourhood of \tilde{K} , which bounds a Riemann surface \tilde{Y} that is diffeomorphic to Y_1 . Moreover, the pattern of \tilde{L} is \mathcal{L} . Then $g_4(\tilde{K}) = g_4(K)$ and $g_4(\tilde{L}) = g_4(L)$ (since the isotopy which moves K to \tilde{K} moves L to a knot in a tubular neighbourhood of \tilde{K} which is isotopic to L and has the same pattern as L and thus as \tilde{L}). Hence,

$$g_4(L) = g(\tilde{Y}) = g(Y_1) = ng(X_1) + g_4(L) = ng_4(K) + g_4(\mathcal{L})$$

The lemma is proved. \square

Added in proof. The proof of Theorem 1 gives immediately also the following variant of the first two statements for the 4-ball Euler characteristics χ_4 of links (defined similarly as the 4-ball genus):

For each smoothly analytic knot $K \subset \partial\mathbb{B}^2$ there is a tubular neighbourhood $N(K) \subset \partial\mathbb{B}^2$ of K such that for each analytic link $L \subset N(K)$ with winding number $n \geq 1$ the inequality $\chi_4(L) \leq n\chi_4(K)$ holds.

Indeed, notice that statement (15) is also true if Y is not connected. One has to choose the algebraic curve C in the proof of statements (1) and (2) to be connected.

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